

# Multimetric continuous model theory

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# MULTIMETRIC CONTINUOUS MODEL THEORY

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ABSTRACT. In this paper, we study metric structures with a finite number of metrics by extending the model theory developed by Ben Yaacov et al. in the monograph *Model theory for metric structures*. We first define a metric structure with finitely many metrics, develop the theory of ultraproducts of multimetric structures, and prove some classical model-theoretic theorems about saturation for structures with multiple metrics. Next, we give a characterization of axiomatizability of certain classes of multimetric structures. Finally, we discuss potential avenues of research regarding structures with multiple metrics.

## 1. MOTIVATIONS AND HISTORY

While first-order logic is well-suited to studying algebraic structures, there have been a number of logics developed which are suited to studying concepts from analysis. We are interested in comparing the expressive power of these logics for analysis.

One logic for probability spaces is the integral logic  $L_{\mathcal{A}f}$ , first developed by Keisler in [8] and Hoover in [7] and recently further developed in [1]. In this logic, integrals are allowed in formulas, with  $\int \tau dx$  binding the variable  $x$ . The set of formulas of  $L_{\mathcal{A}f}$  includes formulas of the form  $\int \tau dx \geq 0$ , which, when interpreted in a natural way, are true or false for a structure. Another logic for probability spaces is the probability logic  $L_{\mathcal{A}P}$ , described by Keisler in [9]. Probability logic is similar to (infinitary) first-order logic, except that the quantifier  $(Px \geq r)$  is used in formulas. As stated in [9], the formula  $(Px \geq r)\varphi(x)$  is interpreted as meaning that the probability of the set  $\{x|\varphi(x)\}$  is at least  $r$ . In both probability and integral logic, theorems from analysis are used to prove logical results.

A logic for Banach spaces was introduced by Henson in the 1970s and developed in papers such as [5]. An introduction to this logic is given in [6]. Nonstandard hulls play an important role in this logic, as does the notion of approximate satisfaction (rather than satisfaction) of a formula. As shown in [6], many important concepts and theorems from first-order model theory have analogues in Banach space logic. We note here that the Banach space  $L^p(\mu)$  ( $\mu$  a measure,  $1 \leq p \leq \infty$ ) is an example of a link between Banach space theory and measure theory, and there has been some interest in the relationship between Banach space logic and the probability logic described previously.

Measure spaces with two measures have been studied in logic before. The idea of studying a logic for structures with two measures was suggested by Keisler in [9]. A particular motivation for this paper is Chapter 6 of [10], which describes biprobability logics for probability spaces with two measures  $\mu_1, \mu_2$  such that one measure is absolutely continuous with respect to the other. In this chapter, the authors modify the probability logic  $L_{\mathcal{A}P}$  developed by Keisler to include two quantifiers

$(P_1\bar{x} \geq r)$  and  $(P_2\bar{x} \geq r)$ . In a similar way as in the single-measure probability logic, the quantifier  $(P_i\bar{x} \geq r)$  can be interpreted as saying that the measure of certain sets with respect to  $\mu_i$  are at least  $r$ . The authors also describe a biprobability logic with two integral operators partly based on the logic  $L_{\mathcal{A}f}$  described in [9], and use the Radon-Nikodym theorem and Fubini's theorem to prove results about existence of models for this biprobability logic.

Recently, a logic for complete metric spaces was developed in [2], inspired by the continuous model theory of Chang and Keisler described in [3] and the Banach space model theory described in [6]. While the continuous model theory in [3] allowed truth values to be taken in any compact Hausdorff space, the model theory for metric structures in [2] restricts truth values to a closed, bounded interval of  $\mathbb{R}$  (with the standard topology). As noted in the introduction of [2], this logic for metric structures can be considered a generalization of first-order logic, since we can make any set into a metric space by equipping it with the discrete metric. Many important concepts from first-order logic (such as definability of sets) have analogues in this logic [2]. More importantly, this logic is well-suited to studying various topics in analysis, such as Hilbert spaces and probability spaces. In this paper, we give a modification of this logic and prove that a number of theorems from first-order logic hold in this modified logic.

## 2. INTRODUCTION

We study metric structures equipped with finitely many metrics  $\{d_1, \dots, d_n\}$  such that for  $1 \leq j, k \leq n$ , the identity map  $\iota_{j,k} : (M, d_j) \rightarrow (M, d_k)$  and its inverse are uniformly continuous. We require that each metric be bounded. To each such structure, we also associate a metric  $\rho$ , where  $\rho = \max\{d_1, \dots, d_n\}$ . It is easily checked that  $\rho$  is a metric.

**Definition 1.** If  $(M, d_1)$  and  $(M, d_2)$  are metric spaces such that the identity map  $\iota : (M, d_1) \rightarrow (M, d_2)$  and its inverse are uniformly continuous, we say that the metrics  $d_1$  and  $d_2$  are *related by uniform continuity*.

**Example.** Suppose  $X$  is a measure space with  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $X$  and  $\mu, \nu$  are finite measures on  $\mathcal{A}$ . Suppose also that  $\mu$  is absolutely continuous with respect to  $\nu$  and vice versa (in other words,  $\mu$  is equivalent to  $\nu$ ). Define an equivalence relation  $\simeq$  on  $\mathcal{A}$  by  $A \simeq B$  if and only if  $\mu(A \Delta B) = 0$ , where  $A \Delta B$  is the symmetric difference of  $A$  and  $B$ . Since  $\mu \ll \nu$  and  $\nu \ll \mu$ ,  $\mu(A \Delta B) = 0$  if and only if  $\nu(A \Delta B) = 0$ . Let  $\mathcal{B} = \mathcal{A} / \simeq$ . The Nikodym metrics  $\rho_\mu$  and  $\rho_\nu$  defined on  $\mathcal{B}$  are related by uniform continuity (see Proposition 19 in Section 18.4 of [11]).

As this example shows, measures induce metrics in a natural way, which provides a motivation for studying metric structures with finitely many metrics.

Throughout this paper, when we refer to a set  $M$  equipped with finitely many metrics  $\{d_1, \dots, d_n\}$ , we will assume that  $d_i$  and  $d_k$  are related by uniform continuity for  $1 \leq i, k \leq n$  and that  $M$  is also equipped with the maximum metric  $\rho$  unless otherwise noted.

We now extend the definition of a modulus of uniform continuity for a function given in [2].

**Definition 2.** Let  $M$  be a metric space equipped with  $n$  metrics  $d_1, \dots, d_n$  and let  $N$  be a metric space equipped with  $n$  metrics  $d'_1, \dots, d'_n$ . Suppose that  $M$  is also

equipped with the maximum metric  $\rho$  and  $N$  is equipped with the maximum metric  $\rho'$ . Let  $f : M \rightarrow N$  be a function. We say that the function  $\Delta_{f,i} : (0, 1] \rightarrow (0, 1]$  is a *modulus of uniform continuity for  $f$  and  $d_i$*  if for all  $\varepsilon \in (0, 1]$  and for all  $a, b \in M$ ,

$$\text{if } d_i(a, b) < \Delta_{f,i}(\varepsilon), \text{ then } d'_i(f(a), f(b)) < \varepsilon \quad (*)$$

Similarly, we say that  $\Delta_{f,\rho} : (0, 1] \rightarrow (0, 1]$  is a modulus of uniform continuity for  $f$  and  $\rho$  if  $(*)$  holds when  $\Delta_{f,i}$  is replaced by  $\Delta_{f,\rho}$ ,  $d_i$  is replaced by  $\rho$  and  $d'_i$  is replaced by  $\rho'$ .

Note that this definition is slightly different from the definition of a modulus of uniform continuity in [2]. In particular, since  $M$  and  $N$  may both have more than one metric, we must decide which metrics on the domain and range should be used in the definition of a modulus of uniform continuity. Since we assume throughout this paper that the metrics  $d_1, \dots, d_n$  on  $M$  are related by uniform continuity and the metrics  $d'_1, \dots, d'_n$  on  $N$  are related by uniform continuity, the order in which we label the metrics on  $M$  and  $N$  does not matter for the definition of modulus of uniform continuity.

We will assume throughout that a modulus of uniform continuity for a function between spaces with multiple metrics is defined as in the previous definition unless otherwise noted.

**Proposition 3.** *If  $M$  is a metric space with finitely many metrics  $\{d_1, \dots, d_n\}$ , then for  $1 \leq k \leq n$ , the metrics  $d_k$  and  $\rho$  are related by uniform continuity.*

*Proof.* It is obvious that the identity map  $\iota : (M, \rho) \rightarrow (M, d_k)$  is uniformly continuous for any  $1 \leq k \leq n$ , since for  $\varepsilon > 0$ ,  $\Delta(\varepsilon) = \varepsilon$  is a modulus of uniform continuity. We now check that for  $1 \leq k \leq n$ , the identity map  $\iota_k : (M, d_k) \rightarrow (M, \rho)$  is uniformly continuous. Let  $\varepsilon > 0$ . We want to show that there exists  $\delta > 0$  such that for all  $x, y \in M$ , if  $d_k(x, y) < \delta$ , then  $\rho(x, y) < \varepsilon$ . Since  $d_k$  and  $d_i$  are related by uniform continuity for all  $1 \leq i \leq n$ , for each  $i = 1, \dots, n$ , there exists  $\alpha_i > 0$  such that for all  $x, y \in M$ , if  $d_k(x, y) < \alpha_i$ , then  $d_i(x, y) < \varepsilon$ . Let  $\delta = \min\{\alpha_1, \dots, \alpha_n, \varepsilon\}$ . Then for all  $x, y \in M$ , if  $d_k(x, y) < \delta$ , then we have  $d_i(x, y) < \varepsilon$  for all  $1 \leq i \leq n$ . Therefore, by definition,  $\rho(x, y) < \varepsilon$ .  $\square$

### 3. MULTIMETRIC STRUCTURES AND SIGNATURES

Let  $M_1, \dots, M_l$  be sets such that for  $1 \leq k \leq l$ ,  $M_i$  is equipped with  $n$  metrics  $d_{1,i}, \dots, d_{n,i}$ . Throughout this paper, we take  $M_1 \times \dots \times M_l$  to be the set equipped with metrics  $\{\hat{d}_1, \dots, \hat{d}_n\}$ , where the metrics are defined as follows. For  $x, y \in M_1 \times \dots \times M_l$ , let  $x = (x_1, \dots, x_l)$  and  $y = (y_1, \dots, y_l)$ . For each  $1 \leq k \leq n$ , we let  $\hat{d}_k(x, y) = \max\{d_{k,1}(x_1, y_1), \dots, d_{k,l}(x_l, y_l)\}$ . Similarly, since each  $M_i$  is also equipped with a maximum metric  $\rho_i$ , we define a metric  $\hat{\rho}$  on  $M_1 \times \dots \times M_l$  by  $\hat{\rho}(x, y) = \max\{\rho_1(x_1, y_1), \dots, \rho_l(x_l, y_l)\}$ . Throughout this paper, unless otherwise noted, a metric  $\hat{d}_k$  or  $\hat{\rho}$  on a product of metric spaces will denote the metric described here.

**Proposition 4.** *If  $\rho_i = \max\{d_{1,i}, \dots, d_{n,i}\}$  on  $M_i$  for  $1 \leq i \leq l$ , then on  $M_1 \times \dots \times M_l$ ,  $\hat{\rho} = \max\{\hat{d}_1, \dots, \hat{d}_n\}$ , where  $\hat{d}$  denotes the product metric defined in the previous paragraph.*

*Proof.* Let  $x, y \in M_1 \times \dots \times M_l$  and suppose  $\max\{\widehat{d}_1(x, y), \dots, \widehat{d}_n(x, y)\} = \widehat{d}_k(x, y)$ . By definition,  $\widehat{d}_k(x, y) = \max\{d_{k,1}(x_1, y_1), \dots, d_{k,l}(x_l, y_l)\}$ . Therefore, there exists  $i$  with  $1 \leq i \leq n$  such that  $d_{k,i}(x_i, y_i) \geq d_{j,m}(x_m, y_m)$  for all  $1 \leq j \leq n$  and  $1 \leq m \leq p$ . By definition of  $\widehat{d}_j$  and  $\widehat{\rho}$ , it is clear that  $\widehat{\rho}(x, y) = \widehat{d}_k(x, y)$ .  $\square$

We now give the definition of predicates, functions, and constants in the metric structure theory, following the definitions given in Section 2 of [2].

**Definition 5.** Let  $M$  be a set equipped with finitely many metrics  $\{d_1, \dots, d_n\}$  such that  $M$  is complete with respect to each metric, and each metric is bounded.

- (1) A *predicate* on  $M$  is a function  $P$  from  $M^n$  (for some  $n \geq 1$ ) into some bounded interval in  $\mathbb{R}$  such that  $P$  is uniformly continuous with respect to each metric on  $M$ .
- (2) A *function* on  $M$  is a function  $f$  from  $M^n$  into  $M$  that is uniformly continuous with respect to each metric on  $M^n$ . In this case, we call  $n$  the arity of the function.

In other words, we associate  $n + 1$  moduli of continuity to each predicate and each function on  $M$ .

Note that the definition of a predicate given above is a generalization of the notion of a relation in first-order logic. An  $n$ -ary relation  $R$  on a set  $A$  is a subset of  $A^n$ . Therefore, we can define a characteristic function  $\chi_R$  on  $A^n$  for  $R$  by  $\chi_R(\bar{a}) = 0$  if  $\bar{a} \notin R$  and  $\chi_R(\bar{a}) = 1$  if  $\bar{a} \in R$ . Thus, a relation in first-order logic is a function from  $A^n$  into  $\{0, 1\}$ . Note that although 0 usually corresponds to the truth value "false", in continuous model theory, we consider a sentence to be true in a structure if it has a value of 0 in that structure.

We now define a multimetric structure, based on the definition of a metric structure in [2].

**Definition 6.** A *multimetric structure*  $\mathcal{M}$  consists of a family  $(R_i : i \in I)$  of predicates on  $M$ , a family  $(F_j : j \in J)$  of functions on  $M$ , and a family  $(a_k : k \in K)$  of elements of  $M$ .

We will denote  $\mathcal{M}$  by

$$\mathcal{M} = (M, R_i, F_j, a_k : i \in I, j \in J, k \in K)$$

Note that any of  $I$ ,  $J$ , or  $K$  can be empty.

A signature in continuous model theory is defined similarly to a signature in first-order model theory. We base our definition of a signature for a multimetric structure on the definition for the signature of a metric structure in [2].

**Definition 7.** A signature  $L$  gives predicate, function, and constant symbols and assigns an arity to each predicate and function symbol as in first-order model theory. We also require that:

- (1)  $L$  provides a positive integer  $n$  which is the number of metrics  $d_1, \dots, d_n$  that each structure has.
- (2) For each predicate symbol  $P$ ,  $L$  provides a closed bounded interval  $I_P$  of real numbers and  $n + 1$  moduli of uniform continuity  $\Delta_{P,1}, \dots, \Delta_{P,n}, \Delta_{P,\rho}$  such that  $P^{\mathcal{M}}$  (the interpretation of the symbol  $P$  in  $\mathcal{M}$ ) takes its values in  $I_P$ ,  $\Delta_{P,i}$  is a modulus of uniform continuity for the metric  $d_i$ , and  $\Delta_{P,\rho}$  is a modulus of uniform continuity for  $\rho$ .

- (3) For each function symbol  $f$ ,  $L$  provides  $n+1$  moduli  $\Delta_{f,1}, \dots, \Delta_{f,n}, \Delta_{f,\rho}$  of uniform continuity for  $f^{\mathcal{M}}$ , where  $\Delta_{f,i}$  is a modulus of uniform continuity for  $f^{\mathcal{M}}$  and  $d_i$  and  $\Delta_{f,\rho}$  is a modulus of uniform continuity for  $f^{\mathcal{M}}$  and  $\rho$ .
- (4)  $L$  provides a nonnegative real number  $D_L$  which is a bound for each of the metric spaces  $(M, d_1), \dots, (M, d_n)$ .

When all of the above requirements are met and the predicate, function, and constant symbols of  $L$  correspond to the predicate, function, and constant symbols of  $\mathcal{M}$ , then we say that  $\mathcal{M}$  is an  $L$ -structure. Throughout this paper, when we refer to a signature  $L$ , we will usually assume that  $D_L = 1$  and  $I_P = [0, 1]$  for each predicate symbol  $P$ .

Following [2], we will treat the logical symbols representing the metrics both as symbols and as the interpretations of those symbols in a metric structure. When we treat these logical symbols as symbols, we will refer to them as metric symbols.

We now give the definition of embedding and isomorphism. These definitions are based on Definition 2.3 in [2].

**Definition 8.** Let  $L$  be a signature for multimetric structures and suppose  $\mathcal{M}, \mathcal{N}$  are  $L$ -structures. An *embedding* from  $\mathcal{M} \rightarrow \mathcal{N}$  is a function  $T : M \rightarrow N$  such that  $T : (M, d_k^{\mathcal{M}}) \rightarrow (N, d_k^{\mathcal{N}})$  is an isometry for  $1 \leq k \leq n$  and  $T$  satisfies the following requirements (taken from [2]):

- (1) Whenever  $f$  is an  $n$ -ary function symbol of  $L$  and  $a_1, \dots, a_n \in M$ , we have
$$f^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = T(f^{\mathcal{M}}(a_1, \dots, a_n))$$
- (2) Whenever  $c$  is a constant symbol  $c$  of  $L$ , we have
$$c^{\mathcal{N}} = T(c^{\mathcal{M}})$$
- (3) Whenever  $P$  is an  $n$ -ary predicate symbol of  $L$  and  $a_1, \dots, a_n \in M$ , we have

$$P^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = P^{\mathcal{M}}(a_1, \dots, a_n)$$

We define an isomorphism to be a surjective embedding. (Note that isometries are always injective, so every embedding is injective.) We say that  $\mathcal{M}$  and  $\mathcal{N}$  are isomorphic and write  $\mathcal{M} \cong \mathcal{N}$  if there exists an isomorphism between  $\mathcal{M}$  and  $\mathcal{N}$ . We say that  $\mathcal{M}$  is a substructure of  $\mathcal{N}$  (written  $\mathcal{M} \subseteq \mathcal{N}$ ) if  $M \subseteq N$  and the inclusion map from  $M$  into  $N$  is an embedding of  $\mathcal{M}$  into  $\mathcal{N}$ .

#### 4. CONSTRUCTING MULTIMETRIC STRUCTURES

In this section, we go through the standard construction of the completion of a metric space in the case that the metric space has more than one metric. As we will see, the requirement that (pseudo)metrics are related by uniform continuity is used to ensure that when we take the completion of a metric space, the resulting complete metric space is the same no matter which metric we use to define the completion.

Recall that a pseudometric  $d_0$  is a real-valued function on a set  $X$  satisfying the properties of a metric except for the property that for all  $x, y \in X$ , if  $d_0(x, y) = 0$ , then  $x = y$ . In other words, there may exist  $x, y \in X$  such that  $d_0(x, y) = 0$ , but  $x \neq y$ .

Let  $M_0$  be a set and  $d_{0,1}, \dots, d_{0,n}$  be pseudometrics on  $M_0$  such that  $d_{0,i}$  and  $d_{0,j}$  are related by uniform continuity for  $1 \leq i, j \leq n$ . Define

$$\rho_0(x, y) = \max\{d_{0,1}(x, y), \dots, d_{0,n}(x, y)\}$$

Then  $\rho_0$  is also a pseudometric. Define an equivalence relation  $E$  on  $M_0$  by  $E(x, y)$  if and only if  $\rho_0(x, y) = 0$ . If  $xEy'$  and  $yEy'$ , then  $\rho_0(x, y) = \rho_0(x', y')$  by the triangle inequality. Let  $M = M_0/E$  and let  $\pi : M_0 \rightarrow M$  be the quotient map. Define  $n+1$  metrics on  $M$ ,  $d_1, \dots, d_n, \rho$ , by taking  $\rho(\pi(x), \pi(y)) = \rho_0(x, y)$  ( $x, y \in M_0$ ) and similarly for each  $d_i$ . Note that  $d_i$  and  $d_j$  are still related by uniform continuity for  $1 \leq i, j \leq n$ . Note also that  $\rho$  is a metric, since  $\rho(\pi(x), \pi(y)) = 0$  if and only if  $\rho_0(x, y) = 0$  if and only if  $\pi(x) = \pi(y)$ . Also note that since the metrics  $d_i, d_j$  are related by uniform continuity for all  $1 \leq i, j \leq n$ , each  $d_i$  is also a metric.

We now define the interpretation of formulas in a prestructure and then define the interpretation of formulas in a multimetric structure.

Let  $\mathcal{M}$  be an  $L$ -prestructure (so that the metric space is not necessarily complete with respect to any of the metrics) with metrics  $d_1, \dots, d_m, \rho$ . We want to take a completion of  $\mathcal{M}$ . We do this by considering equivalence classes of Cauchy sequences that are Cauchy with respect to  $\rho$ . Denote this set of Cauchy sequences by  $C$ . We define a relation on  $C$  by  $(x_n) \sim (y_n)$  if and only if  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$ . Let  $\bar{X}$  be the set of equivalence classes under the  $\sim$  relation. We now define new metrics on  $\bar{X}$  by  $\bar{\rho}([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} \rho(x_n, y_n)$  and similarly for  $d_1, \dots, d_m$ . It can be checked that  $\rho$  is a metric. (Note that the limit exists because the sequence  $\rho(x_n, y_n)$  is Cauchy.) We denote the space of equivalence classes of  $C$  by  $C'$ .

It should be noted that a sequence  $\{x_n\}$  in  $M$  is Cauchy with respect to  $d_i$  if and only if it's Cauchy with respect to  $d_j$  for any  $1 \leq i, j \leq m$ , and a sequence  $\{x_n\}$  is Cauchy with respect to  $d_1, \dots, d_m$  if and only if it's Cauchy with respect to  $\rho$ . In particular, this says that for any sequences  $(x_n), (y_n)$  that are Cauchy for  $\rho$ ,  $\bar{d}_i([(x_n)], [(y_n)])$  exists.

**Lemma 9.** *Let  $(x_n), (y_n)$  be Cauchy sequences in  $M$ . Then  $\lim_{n \rightarrow \infty} d_i(x_n, y_n) = 0$  if and only if  $\lim_{n \rightarrow \infty} d_j(x_n, y_n) = 0$  for all  $1 \leq i, j \leq m$ , and  $\lim_{n \rightarrow \infty} d_i(x_n, y_n) = 0$  if and only if  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$  for  $1 \leq i \leq m$ .*

This says that it doesn't matter which metric we use to define  $C'$ .

*Proof.* Fix  $i, j$  with  $1 \leq i, j \leq m$ . For the first "if and only if" statement in the "only if" direction, let  $\varepsilon > 0$  and let  $\delta > 0$  be such that for all  $a, b \in M$ , if  $d_i(a, b) < \delta$  then  $d_j(a, b) < \varepsilon$ . Now let  $N \in \mathbb{N}$  be such that if  $n \geq N$ , then  $d_i(x_n, y_n) < \delta$  (such an  $N$  exists because  $\lim_{n \rightarrow \infty} d_i(x_n, y_n) = 0$ ). Now if  $n \geq N$ , then  $d_i(x_n, y_n) < \delta$ , so  $d_j(x_n, y_n) < \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary,  $\lim_{n \rightarrow \infty} d_j(x_n, y_n) = 0$ . A similar proof shows the other direction of this statement.

The proof of the second statement, using the first statement, is clear.  $\square$

**Lemma 10.** *Let  $\bar{d}_i$ ,  $1 \leq i \leq m$ , be the function defined on  $C'$  by  $\bar{d}_i([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d_i(x_n, y_n)$ . Then  $\bar{d}_i$  is a metric on  $C'$ .*

*Proof.* The only thing we need to check is that  $\bar{d}_i([(x_n)], [(y_n)]) = 0$  if and only if  $[(x_n)] = [(y_n)]$ . If  $[(x_n)] = [(y_n)]$ , then by definition,  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$ . By what we showed in the previous theorem, this implies that  $\lim_{n \rightarrow \infty} d_i(x_n, y_n) = 0$ , so by definition,  $\bar{d}_i([(x_n)], [(y_n)]) = 0$ .



Now suppose  $\bar{d}_i([(x_n)], [(y_n)]) = 0$ . Then by definition,  $\lim_{n \rightarrow \infty} d_i(x_n, y_n) = 0$ . Therefore, we have  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$  by the previous theorem, and by definition of the equivalence relation,  $(x_n) \sim (y_n)$ , so  $[(x_n)] = [(y_n)]$ .  $\square$

**Lemma 11.** *The metrics  $\bar{d}_i, \bar{d}_j$  are related by uniform continuity for  $1 \leq i, j \leq m$ . The metrics  $\bar{d}_i$  and  $\bar{\rho}$  are related by uniform continuity for  $1 \leq i \leq m$ .*

*Proof.* Let  $\varepsilon > 0$ . Since  $d_i, d_j$  are related by uniform continuity, there exists  $\delta > 0$  such that for all  $x, y \in M$ , if  $d_i(x, y) < \delta$ , then  $d_j(x, y) < \varepsilon$ . If  $\bar{d}_i(x, y) < \delta$ , then there exists  $K$  such that if  $n \geq K$ , then  $d_i(x_n, y_n) < \delta$ . Therefore, if  $n \geq K$ , we also have  $d_j(x_n, y_n) < \varepsilon$ , so we must have  $\bar{d}_j(x, y) < \varepsilon$  by definition.  $\square$

The proof of the second statement is similar.  $\square$

Note that we still have  $\bar{\rho} = \max\{\bar{d}_1, \dots, \bar{d}_m\}$ . Also note that  $\widehat{\bar{d}}_i$  and  $\widehat{\bar{d}}_j$  are related by uniform continuity for  $1 \leq i, j \leq m$ , and that  $\widehat{\bar{\rho}}$  and  $\widehat{\bar{d}}_i$  are related by uniform continuity for  $1 \leq i \leq m$ .

Note also that  $C'$  is the completion of  $M$  and that  $M$  is dense in  $C'$ . In other words,  $(C', \bar{d}_i)$  is the completion of  $(M, d_i)$  for  $1 \leq i \leq m$ ,  $(C', \bar{\rho})$  is the completion of  $(M, \rho)$ , and  $M$  is dense in  $C'$  in each of these cases. The proof for each metric is the same as the proof in the single-metric case, since it does not matter which metric we use to define the equivalence relation on  $C'$ . An outline of this proof is given in Exercise 49 of Section 9.4 of [11]. This exercise also shows that the function  $\varphi : M \rightarrow C'$  given by  $\varphi(a) = [(a)]$  (the equivalence class of the constant sequence  $a$ ) is an isometry for each of the metrics.

**Lemma 12.** *If  $C'$  is the completion of  $M$  given above, then  $(C')^n$  is a completion of  $M^n$  for each metric.*

*Proof.* Let  $\psi : M^n \rightarrow (C')^n$  be given by  $\psi(a_1, \dots, a_n) = ([a_1]), \dots, [(a_n)]$  (where on the right hand side, the equivalence class  $[(a_j)]$  is the equivalence class of the constant sequence  $(a_j)_{i \in \mathbb{N}}$ ). For any  $1 \leq i \leq m$ , we have

$$\begin{aligned} \widehat{\bar{d}}_i((a_1, \dots, a_n), (b_1, \dots, b_n)) &= \max\{d_i(a_1, b_1), \dots, d_i(a_n, b_n)\} \\ &= \max\{\bar{d}_i([(a_1)], [(b_1)]), \dots, \bar{d}_i([(a_n)], [(b_n)])\} \\ &= \widehat{\bar{d}}_i([(a_1)], \dots, [(a_n)], [(b_1)], \dots, [(b_n)]) \end{aligned}$$

by the fact that  $\varphi$  is an isometry. A similar proof shows that the above equations hold for  $\widehat{\bar{\rho}}$  and  $\widehat{\bar{p}}$ . Therefore,  $\psi$  is an isometry for each metric on  $M^n$ .

We now show that  $(C')^n$  is complete with respect to  $\widehat{\bar{d}}_j$  for  $1 \leq j \leq n$  and with respect to  $\widehat{\bar{\rho}}$ . Let  $([a_1^i], \dots, [a_n^i])_{i \in \mathbb{N}}$  be a Cauchy sequence in  $(C')^n$  with respect to  $\widehat{\bar{d}}_j$ . (Note that the property of being a Cauchy sequence in  $(C')^n$  does not depend on which metric we use on  $(C')^n$ , since  $\widehat{\bar{d}}_i$  and  $\widehat{\bar{d}}_j$  ( $1 \leq i, j \leq m$ ) are still related by uniform continuity and  $\widehat{\bar{\rho}}$  is still the maximum of these metrics.) Then the sequences of the form  $\{[a_k^i]\}_{i \in \mathbb{N}}$  ( $1 \leq k \leq n$ ) are Cauchy sequences in  $C'$  by definition of the metric on  $(C')^n$  and the definition of Cauchy sequence. Since  $C'$  is complete with respect to all induced metrics, each of these Cauchy sequences converges to an element, say  $[b_i]$  for the  $i$ th Cauchy sequence, in  $C'$  with respect to  $\bar{d}_j$ . It is easy

to see that the sequence  $([a_1^i], \dots, [a_n^i])_{i \in \mathbb{N}}$  converges to  $([b_1], \dots, [b_n]) \in (C')^n$  for  $\widehat{d}_j$ , and thus that  $(C')^n$  is complete for each metric, including  $\widehat{\rho}$ .

We now show that  $\psi(M^n)$  is dense in  $((C')^n$  for each metric. Let  $([a_1], \dots, [a_n]) \in (C')^n$  and let  $\varepsilon > 0$ . Let  $1 \leq j \leq m$ . Since  $\varphi(M)$  is dense in  $C'$ , there exists  $[b_i] \in \varphi(M)$  such that for  $1 \leq i \leq n$ ,  $\bar{d}_j([a_i], [b_i]) < \varepsilon$ . Consider  $([b_1], \dots, [b_n]) \in (C')^n$ . Then by definition of the metric on  $(C')^n$ , we have  $\widehat{d}_j(( [a_1], \dots, [a_n] ), ([b_1], \dots, [b_n])) < \varepsilon$ . Thus,  $\psi(M^n)$  is dense in  $(C')^n$  for  $\widehat{d}_j$ . A similar proof shows the same thing for  $\widehat{\rho}$ .  $\square$

From here on, we denote the completion of  $M$  by  $N$  (note that  $N$  is what we previously called  $C'$ ).

**Lemma 13.** *Let  $f : M^n \rightarrow M$  be uniformly continuous with respect to  $\widehat{d}_j$  for  $1 \leq j \leq n$  and  $\widehat{\rho}$ . Let  $g : \psi(M^n) \rightarrow N$  be the function defined by  $g(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(f(x_1, \dots, x_n))$ , where  $\varphi : M \rightarrow N$  is as defined above and  $x_1, \dots, x_n \in M$ . Then  $g$  is uniformly continuous with respect to all metrics.*

*Proof.* Let  $\varepsilon > 0$ . We first show that this holds for  $\widehat{d}_j$ ,  $1 \leq j \leq m$ . By the uniform continuity of  $f$ , there exists  $\delta > 0$  such that for all  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n) \in M^n$ , if  $\widehat{d}_j(x, y) < \delta$ , then  $d_j(f(x), f(y)) < \varepsilon$ . Now suppose  $\widehat{d}_j((\varphi(x_1), \dots, \varphi(x_n)), (\varphi(y_1), \dots, \varphi(y_n))) < \delta$ . Then by definition,

$$\max\{\bar{d}_j(\varphi(x_1), \varphi(y_1)), \dots, \bar{d}_j(\varphi(x_n), \varphi(y_n))\} < \delta$$

Since  $\varphi$  is an isometry for each metric, we have  $\max\{d_j(x_1, y_1), \dots, d_j(x_n, y_n)\} < \delta$ . Since this is the definition of  $\widehat{d}_j$ , we also have  $d_j(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) < \varepsilon$ . Again, since  $\varphi$  is an isometry, we have  $\bar{d}_j(\varphi(f(x_1, \dots, x_n)), \varphi(f(y_1, \dots, y_n))) < \varepsilon$ . So by definition of  $g$ ,  $\bar{d}_j(g(\varphi(x_1), \dots, \varphi(x_n)), g(\varphi(y_1), \dots, \varphi(y_n))) < \varepsilon$ . This was what we wanted. A similar proof shows that  $g$  is uniformly continuous with respect to  $\widehat{\rho}$ .  $\square$

**Lemma 14.** *For each  $x \in N^n$  and for  $1 \leq i \leq m$ , let  $\{x_k^i\}_{k \in \mathbb{N}}$  be a sequence in  $\psi(M^n)$  that converges to  $x$  with respect to  $\widehat{d}_i$ . Let  $\{y_k\}_{k \in \mathbb{N}}$  be a sequence in  $\psi(M^n)$  that converges to  $x$  with respect to  $\widehat{\rho}$ . Then  $\lim_{k \rightarrow \infty} g(x_k^1) = \dots = \lim_{k \rightarrow \infty} g(x_k^m) = \lim_{k \rightarrow \infty} g(y_k)$ , where the limit  $\lim_{k \rightarrow \infty} g(x_k^i)$  is taken with respect to  $\bar{d}_i$  and the limit  $\lim_{k \rightarrow \infty} g(y_k)$  is taken with respect to  $\bar{\rho}$ .*

This lemma states that the value of this limit does not depend on which metric we use. Note that in the last line of the theorem, each limit exists because the sequences  $\{x_k^i\}_{k \in \mathbb{N}}$ ,  $1 \leq i \leq m$ , and  $\{y_k\}$  are convergent sequences in  $\psi(M^n)$ , so they are Cauchy. Since  $g$  is uniformly continuous with respect to each metric, the sequences of the form  $\{g(x_k^i)\}$  and the sequence  $\{g(y_k)\}$  are also Cauchy for each metric. So since  $N$  is complete with respect to all metrics, the sequences  $\{g(x_k^i)\}$  and  $\{g(y_k)\}$  converge in  $N$ .

*Proof.* For the equality of the limits of the  $g(x_k^i)$ 's, let  $\varepsilon > 0$  and fix  $i, j \leq m$ . Let  $L_i = \lim_{n \rightarrow \infty} g(x_n^i)$ , where the limit is taken with respect to  $\bar{d}_i$ . Note that  $L_i \in N$ . We want to show that there exists  $K \in \mathbb{N}$  such that if  $k \geq K$ , then  $\bar{d}_j(g(x_k^j), L_i) < \varepsilon$ .

Let  $\delta_1 > 0$  be such that if  $a, b \in N$  and  $\bar{d}_i(a, b) < \delta_1$ , then  $\bar{d}_j(a, b) < \varepsilon/2$ . Now let  $\delta_2 > 0$  be such that for all  $a, b \in \psi(M^n)$  with  $\widehat{\bar{d}}_j(a, b) < \delta_2$ , we have  $\bar{d}_j(g(a), g(b)) < \varepsilon/2$ . Such a  $\delta_2$  exists because  $g$  is uniformly continuous on  $\psi(M^n)$ . Let  $K_1 \in \mathbb{N}$  be such that if  $k \geq K_1$ , then  $\bar{d}_i(g(x_k^i), L_i) < \delta_1$ . Let  $K_2 \in \mathbb{N}$  be such that if  $k \geq K_2$ , then  $\widehat{\bar{d}}_j(x_k^i, x_k^j) < \delta_2$ . (There exists  $M_1$  such that if  $m \geq M_1$ , then  $\widehat{\bar{d}}_i(x_m^i, x) < \alpha$ , where  $\alpha > 0$  is such that for all  $a, b \in N^n$ , if  $\widehat{\bar{d}}_i(a, b) < \alpha$ , then  $\widehat{\bar{d}}_j(a, b) < \varepsilon/2$ . Also, there exists  $M_2$  such that if  $m \geq M_2$ , then  $\widehat{\bar{d}}_j(x_m^j, x) < \varepsilon/2$ . Now let  $K_2 = \max\{M_1, M_2\}$ . If  $k \geq K_2$ , then  $\widehat{\bar{d}}_i(x_k^i, x) < \alpha$ , so  $\widehat{\bar{d}}_j(x_k^i, x) < \varepsilon/2$ , and  $\widehat{\bar{d}}_j(x_k^j, x) < \varepsilon/2$ . Thus,  $\widehat{\bar{d}}_j(x_k^i, x_k^j) \leq \widehat{\bar{d}}_j(x_k^i, x) + \widehat{\bar{d}}_j(x, x_k^j) < \varepsilon$ .) Now let  $K = \max\{K_1, K_2\}$ . If  $k \geq K$ , then  $\bar{d}_i(g(x_k^i), L_i) < \delta_1$ , so  $\bar{d}_j(g(x_k^i), L_i) < \varepsilon/2$ . Also, if  $k \geq K$ , then  $\widehat{\bar{d}}_j(x_k^i, x_k^j) < \delta_2$ , so  $\bar{d}_j(g(x_k^i), g(x_k^j)) < \varepsilon/2$ . Thus,  $\bar{d}_j(g(x_k^j), L_i) \leq \bar{d}_j(g(x_k^i), g(x_k^j)) + \bar{d}_j(g(x_k^i), L_i) < \varepsilon$ , and  $g(x_k^j) \rightarrow L_i$  with respect to  $\bar{d}_j$ .

Since  $\bar{d}_i$  and  $\bar{d}_j$  are related by uniform continuity and  $\bar{d}_i$  and  $\bar{p}$  are related by uniform continuity (for  $1 \leq i, j \leq m$ ), this proof works for the other metrics as well.  $\square$

**Theorem 15.** *For  $x \in N^n$ , let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence in  $\psi(M^n)$  that converges to  $x$ . Define a function  $\bar{g} : N^n \rightarrow N$  by  $\bar{g}(x) = \lim_{n \rightarrow \infty} g(x_n)$ . The function  $\bar{g}$  is well-defined, uniquely extends  $g$ , and is uniformly continuous for each metric on  $N^n$ .*

*Proof.* A proof is outlined in Exercise 44 in Section 9.4 of [11].  $\square$

In a similar manner as above, we can extend predicates  $P : M^n \rightarrow I_P$  to predicates from  $N^n$  into  $I_P$ .

We now define the values of predicate, function, and constant symbols on  $N$  (the completion of  $M$ ) as in [2]. From this, we obtain an  $L$ -structure  $\mathcal{N}$ . In a similar manner as [2], we only use the term  $L$ -structure (rather than  $L$ -prestructure) when the set  $N$  is complete with respect to all metrics.

## 5. FORMULAS

We inductively define terms, atomic formulas, and formulas of a signature  $L$  in a similar manner as in [2]. The only item of note here is that we consider  $d(t_1, t_2)$  to be an atomic formula for any metric symbol  $d$  of  $L$  and terms  $t_1, t_2$ .

**Notation.** Let  $L$  be a signature and let  $\mathcal{M}$  be an  $L$ -structure. Let  $A \subseteq M$ . As in [2], we will denote the signature  $L$  together with new constant symbols  $\{c(a)\}_{a \in A}$  by  $L(A)$ . We can extend  $\mathcal{M}$  to an  $L(A)$ -structure by interpreting  $c(a)$  by  $a$ , and we will denote this extended  $L(A)$ -structure by  $(\mathcal{M}, a)_{a \in A}$  or  $\mathcal{M}_A$ .

We follow Definitions 3.3 and 3.4 in [2] in defining the value of sentences and formulas. The value of an  $L$ -sentence  $\varphi$  in an  $L$ -prestructure  $\mathcal{M}$ , denoted  $\varphi^{\mathcal{M}}$ , is defined inductively as in Definition 3.3. For an  $L(M)$ -formula  $\varphi(x_1, \dots, x_n)$ , we consider  $\varphi^{\mathcal{M}} : M^n \rightarrow [0, 1]$  to be the function defined in Definition 3.4.

The next theorem is similar to Theorem 3.5 in [2]. However, in this version of the theorem, the moduli of uniform continuity depend on the particular structure  $\mathcal{M}$ .

**Theorem 16.** Let  $t(x_1, \dots, x_n)$  be an  $L$ -term and  $\varphi(x_1, \dots, x_n)$  be an  $L$ -formula, where the number of metrics provided by  $L$  is  $m$ . Let  $\mathcal{M}$  be an  $L$ -prestructure equipped with metrics  $d_1, \dots, d_m, \rho$ . Then there exist  $m+1$  functions  $\Delta_{t,1}, \dots, \Delta_{t,m}, \Delta_{t,\rho}$  and  $m+1$  functions  $\Delta_{\varphi,1}, \dots, \Delta_{\varphi,m}, \Delta_{\varphi,\rho}$  such that  $\Delta_{t,1}, \dots, \Delta_{t,m}, \Delta_{t,\rho}$  are moduli of uniform continuity for  $t^{\mathcal{M}} : M^n \rightarrow M$  and  $\Delta_{\varphi,1}, \dots, \Delta_{\varphi,m}, \Delta_{\varphi,\rho}$  are moduli of uniform continuity for  $\varphi^{\mathcal{M}} : M^n \rightarrow [0, 1]$ .

*Proof.* The proof of this theorem is the same as the proof of Theorem 3.5 in every step of the induction except for the metric step. Therefore, we check this step of the induction.

Suppose that each term  $t$  has  $m+1$  moduli of continuity  $\Delta_{t,1}, \dots, \Delta_{t,m}, \Delta_{t,\rho}$ . Fix  $i$  and  $j$  with  $1 \leq i, j \leq m$ . We want to show that for  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\bar{a}, \bar{b} \in M^n$ , if  $\widehat{d}_j(\bar{a}, \bar{b}) < \delta$ , then  $|d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{a})) - d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), t_2^{\mathcal{M}}(\bar{b}))| < \varepsilon$ . Since  $d_i$  and  $d_j$  are related by uniform continuity, there exists  $\delta_1$  such that for all  $a, b \in M$ , if  $d_j(a, b) < \delta_1$ , then  $d_i(a, b) < \Delta_{t_1, i}(\varepsilon/2)$ . There also exists  $\delta_2$  such that for all  $a, b \in M$ , if  $d_j(a, b) < \delta_2$ , then  $d_i(a, b) < \Delta_{t_2, i}(\varepsilon/2)$ . Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then for  $\bar{a} = (a_1, \dots, a_n)$  and  $\bar{b} = (b_1, \dots, b_n) \in M^n$ , if  $\widehat{d}_j(\bar{a}, \bar{b}) < \delta$ , then by definition,  $\max\{d_j(a_1, b_1), \dots, d_j(a_n, b_n)\} < \delta$ . So for all  $1 \leq k \leq n$ ,  $d_j(a_k, b_k) < \delta$ , and so  $\widehat{d}_i(\bar{a}, \bar{a}) < \Delta_{t_1, i}(\varepsilon/2)$ . Therefore,  $d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_1^{\mathcal{M}}(\bar{b})) < \varepsilon/2$ . By similar reasoning,  $\widehat{d}_i(\bar{a}, \bar{b}) < \Delta_{t_2, i}(\varepsilon/2)$ , so  $d_i^{\mathcal{M}}(t_2^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{b})) < \varepsilon/2$ . Therefore,

$$\begin{aligned} & |d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{a})) - d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), t_2^{\mathcal{M}}(\bar{b}))| \\ &= |d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{a})) - d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{b})) + d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{b})) - d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), t_2^{\mathcal{M}}(\bar{b}))| \\ &\leq |d_i^{\mathcal{M}}(t_2^{\mathcal{M}}(\bar{a}), t_2^{\mathcal{M}}(\bar{b}))| + |d_i^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), t_1^{\mathcal{M}}(\bar{b}))| \\ &< \varepsilon \end{aligned}$$

Since each metric  $d_1, \dots, d_m$  is related to  $\rho$  by uniform continuity, a similar proof shows that  $d_i(t_1(\bar{x}), t_2(\bar{y}))$  has a modulus of uniform continuity for  $\widehat{\rho}$  and that  $\rho(t_1(\bar{x}), t_2(\bar{y}))$  has  $m+1$  moduli of uniform continuity.  $\square$

Theorem 3.7 in [2] also holds in the multimetric case, and the proof is the same.

We now give the definition of an  $L$ -condition and a closed  $L$ -condition in the multimetric case. These definitions are the same as in the single-metric case. The following definition is taken from Section 3 of [2].

**Definition 17.** An  $L$ -condition  $E$  is a formal expression of the form  $\varphi = 0$ , where  $\varphi$  is an  $L$ -formula. If  $\varphi$  is a sentence, we say that  $E$  is *closed*.

As in Remark 3.13 of [2], we define the following abbreviations for  $L$ -conditions.

**Notation.** Let  $\varphi, \psi$  be  $L$ -formulas.

- (1) We abbreviate the condition  $|\varphi - \psi| = 0$  by  $\varphi = \psi$ . In particular, since any  $r \in [0, 1]$  can be considered to be a constant function, we consider  $\varphi = r$  as a condition.
- (2) Let  $\div : [0, 1]^2 \rightarrow [0, 1]$  be the function defined by  $x \div y = \max\{x - y, 0\}$ . Note that  $\div$  is a connective.
- (3) We abbreviate the condition  $\varphi \div \psi = 0$  by  $\varphi \leq \psi$  or  $\psi \geq \varphi$ . Again, we consider  $\varphi \leq r$  to be a condition for  $r \in [0, 1]$ .

Next, we give some notation. The following is Definition 5.9 of [2].

**Notation.** Let  $\Sigma$  be a set of  $L$ -conditions. Then  $\Sigma^+$  is the set of conditions  $\varphi \leq 1/n$  such that  $\varphi = 0$  is a condition in  $\Sigma$  and  $n \in \mathbb{N}$  with  $n \geq 1$ .

## 6. MODEL-THEORETIC DEFINITIONS

The definition of a theory and a model of a theory are the same as in Definition 4.1 in [2]. The metric structure definitions are similar to the definitions of these terms in first-order logic. The definitions of elementary equivalence, elementary substructures/extensions, elementary maps, and elementary embeddings are the same as in Definition 4.3 in [2].

## 7. ULTRAPRODUCTS

We first recall the definition of an ultrafilter on a set  $I$  and the definition of a filter generated by a set. Let  $S(I)$  denote the power set of  $I$ .

**Definition 18.** A *filter*  $D$  on  $I$  is a set  $D \subset S(I)$  such that:

- (1)  $I \in D$
- (2) If  $X \in D$  and  $Y \in D$ , then  $X \cap Y \in D$
- (3) If  $X \in D$  and  $X \subset Z$  with  $Z \in S(I)$ , then  $Z \in D$

If in addition  $D$  satisfies the property that for all  $Z \in S(I)$ , either  $Z \in D$  or  $I \setminus Z \in D$ , then we say that  $D$  is an *ultrafilter*.

**Definition 19.** Let  $E \subset S(I)$ . The *filter generated by  $E$*  is the set

$$D = \bigcap \{E \subset F : F \text{ a filter on } I\}$$

We now introduce some notation.

**Notation.** For a family  $(x_i)_{i \in I}$  of elements of a topological space  $X$ ,  $x \in X$ , and  $D$  an ultrafilter on  $I$ , we write

$$\lim_{i,D} x_i = x$$

if  $\{i \in I : x_i \in U\} \in D$  for every neighborhood  $U$  of  $x$ .

This is the definition of  $D$ -limit given in Section 5 of [2].

We now give the definition of an ultraproduct of multimetric structures, each with the same number of metrics. The ultraproduct of multimetric structures should be a multimetric structure, and the metric structure version of Łoś's theorem (Theorem 5.4 in [2]) should hold in the multimetric case as well. It is natural to define ultraproducts by taking the quotient of a direct product by an equivalence relation defined by one of the metrics on the direct product. In order to make Łoś's theorem hold, we must define the metrics on an ultraproduct of multimetric structures as described below. However, as the following counterexample shows, unless we restrict the classes of models that we may take ultraproducts of, the ultraproduct of multimetric structures is not necessarily a multimetric structure.

### 7.1. First definition of ultraproducts.

Let  $(M_i, d_{1,i}, \dots, d_{m,i}, \rho_i : i \in I)$  be a family of bounded multimetric spaces, with all metrics bounded by the real number  $K$ . Let  $D$  be an ultrafilter on  $I$ . For  $1 \leq k \leq m$ , define a function  $d_k$  on  $\prod_{i \in I} M_i$  by  $d_k(x, y) = \lim_{i, D} d_{k,i}(x_i, y_i)$ , where  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$ . Similarly, we define a function  $\rho$  on  $\prod_{i \in I} M_i$  by  $\rho(x, y) = \lim_{i, D} \rho(x_i, y_i)$ . Note that since  $[0, K]$  is compact and Hausdorff, by the comment on page 24 of [2], these ultrafilter limits exist and are unique.

We now define an equivalence relation on  $\prod_{i \in I} M_i$  by  $x \sim_\rho y$  if and only if  $\rho(x, y) = 0$ . As in the single-metric case, described in [2],  $\rho$  induces a metric  $\rho'$  on the quotient space  $\prod_{i \in I} M_i / \sim_\rho$ . We consider  $\prod_{i \in I} M_i / \sim_\rho$  to be the ultraproduct of the family  $((M_i, d_{1,i}, \dots, d_{m,i}, \rho_i) : i \in I)$ . Therefore, as in the single-metric case,  $(\prod_{i \in I} M_i / \sim_\rho, \rho')$  is a metric space.

We now define functions  $d'_k$  ( $1 \leq k \leq m$ ) on the ultraproduct by

$$d'_k(((x_i)_{i \in I})_D, ((y_i)_{i \in I})_D) = d_k((x_i)_{i \in I}, (y_i)_{i \in I})$$

However, these functions are not necessarily metrics. Consider the following counterexample.

**Example 20.** For  $n \in \mathbb{N}$  with  $n \geq 1$ , let  $X_n = [0, 1]$  and let  $d_{1,n}$  and  $d_{2,n}$  be the metrics on  $X_n$  defined by  $d_{1,n}(x, y) = |x - y|$ ,  $d_{2,n}(x, y) = |x^n - y^n|$  for  $x, y \in X_n$ . Let  $\rho_n = \max\{d_{1,n}, d_{2,n}\}$ . Note that for all  $n \geq 1$ ,  $d_{1,n}$  and  $d_{2,n}$  are related by uniform continuity. Let  $D$  be a nonprincipal ultrafilter on  $\mathbb{N}$ . Let  $\alpha = ((1/2)_{n \in \mathbb{N}})_D$  and let  $\beta = ((1/4)_{n \in \mathbb{N}})_D$ . Then  $d'_2(\alpha, \beta) = 0$ , but  $\alpha \neq \beta$ . This is because  $\rho'(\alpha, \beta) = \lim_{n, D} \rho_n(1/2, 1/4) = \lim_{n, D} \max\{1/4, |(1/2)^n - (1/4)^n|\} = 1/4$ , and  $\rho'$  is still a metric on the ultraproduct. Therefore,  $d'_2$  is not a metric.

It should also be noted that if we try to take the quotient of the direct product  $\prod_{i \in I} M_i$  by an equivalence relation defined by one of the metrics other than  $\rho$ , we may get different equivalence classes. The following is an example of this.

**Example 21.** Let  $X_n, d_{1,n}, d_{2,n}$  be as in the previous example, and let  $D$  be a nonprincipal ultrafilter on  $\mathbb{N}$ . If we define  $x \sim_{d_1} y$  to mean  $d_1(x, y) = 0$  and  $x \sim_{d_2} y$  to mean  $d_2(x, y) = 0$ , we will obtain different equivalence classes when we take the quotient of  $\prod_{i \in I} M_i$  by  $\sim_{d_1}$  versus  $\sim_{d_2}$ . For example, when  $x = ((1/2))_{n \in \mathbb{N}}$  and  $y = ((1/4))_{n \in \mathbb{N}}$ ,  $x \sim_{d_2} y$ , but we do not have  $x \sim_{d_1} y$ .

Therefore, in order to obtain a meaningful definition of ultraproduct, we restrict the classes of structures of which we can take the ultraproduct.

## 7.2. Second definition of ultraproducts.

**Definition 22.** Let  $(M_i, d_{1,i}, \dots, d_{m,i}, \rho_i)$  be a family of multimetric structures such that the family of identity maps  $\varphi_i^{k,j} : (M_i, d_{k,i}) \rightarrow (M_i, d_{j,i})$ , where  $1 \leq k, j \leq m$ , satisfies the following property: For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $i \in I$ , all  $k, j$  with  $1 \leq k, j \leq m$ , and all  $x, y \in M_i$ ,  $d_{k,i}(x, y) < \delta$  implies  $d_{j,i}(x, y) < \varepsilon$ . We say that such a family of metric structures is *uniformly equicontinuous*.

Note that the family  $((X_n, d_{1,n}, d_{2,n}) : n \geq 1)$  discussed in the previous section is not uniformly equicontinuous.

Let  $(M_i, d_{1,i}, \dots, d_{m,i}, \rho_i : i \in I)$  be a uniformly equicontinuous family of bounded spaces with all metrics bounded by the real number  $K$ . Let  $D$  be an ultrafilter on  $I$ .

For  $1 \leq k \leq m$ , define a function  $d_k$  on  $\prod_{i \in I} M_i$  by  $d_k(x, y) = \lim_{i, D} d_{k,i}(x_i, y_i)$ , where  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$ , and similarly  $\rho$ . As before, since  $[0, K]$  is compact and Hausdorff, these ultrafilter limits exist and are unique.

We now define the ultraproduct of a uniformly equicontinuous family of (bounded) metric structures. We define an equivalence relation  $\sim$  on  $\prod_{i \in I} M_i$  by  $x \sim y$  if and only if  $\rho(x, y) = 0$ . As before, this equivalence relation induces  $m + 1$  functions  $d'_1, \dots, d'_m, \rho'$  on  $\prod_{i \in I} M_i / \sim$ . Since we defined  $\sim$  with  $\rho$ , as before,  $\rho'$  is a metric on  $\prod_{i \in I} M_i / \sim$ . We will denote the equivalence class of  $x \in \prod_{i \in I} M_i$  with respect to  $\sim$  by  $[x]$ .

Note that with this definition, we can always take ultrapowers of multimetric structures, but it is not possible to take ultraproducts of arbitrary classes of multimetric structures.

**Lemma 23.** *For  $1 \leq j, k \leq m$ , the functions  $d'_j$  and  $d'_k$  are related by uniform continuity.*

*Proof.* Fix  $j, k$  with  $1 \leq j, k \leq m$ . Let  $\varepsilon > 0$ . We want to show that there exists  $\alpha > 0$  such that for all  $[x], [y] \in \prod_{i \in I} M_i / \sim$ , if  $d'_j([x], [y]) < \alpha$ , then  $d'_k([x], [y]) \leq \varepsilon$ . Since the family  $(\mathcal{M}_i : i \in I)$  is uniformly equicontinuous, there exists  $\delta > 0$  such that for all  $i \in I$  and all  $a, b \in M_i$ ,  $d_{j,i}(a, b) < \delta$  implies  $d_{k,i}(a, b) < \varepsilon$ .

Now suppose  $d'_j([x], [y]) < \delta$ . Then  $\lim_{i, D} d_{j,i}(x_i, y_i) < \delta$  by definition. Suppose for a contradiction that  $\lim_{i, D} d_{k,i}(x_i, y_i) > \varepsilon$ . Let  $\alpha > 0$  be such that  $\varepsilon + \alpha < \lim_{i, D} d_{k,i}(x_i, y_i)$ . By definition of  $D$ -limit, we must have  $\{i \in I : d_{k,i}(x_i, y_i) \in B(\lim_{i, D} d_{k,i}(x_i, y_i), \alpha)\} \in D$ . (Here  $B(x, \gamma) = \{y \in \mathbb{R} : |x - y| < \gamma\}$  for  $x, \gamma \in \mathbb{R}$ .) Since  $D$  is an ultrafilter, we also have  $\{i \in I : d_{k,i}(x_i, y_i) > \varepsilon\} \in D$ . Let  $\beta > 0$  be such that  $\lim_{i, D} d_{j,i}(x_i, y_i) < \delta - \beta$ . Then  $\{i \in I : d_{j,i}(x_i, y_i) \in B(\lim_{i, D} d_{j,i}(x_i, y_i), \beta)\} \in D$ . But for every  $i$  in this set, we have  $d_{k,i}(x_i, y_i) < \varepsilon$  by the fact that  $(\mathcal{M}_i : i \in I)$  is uniformly equicontinuous. So  $\{i \in I : d_{k,i}(x_i, y_i) < \varepsilon\} \in D$ , contradiction. Therefore, if  $d'_j([x], [y]) < \delta$ , then  $d'_k([x], [y]) \leq \varepsilon$ .  $\square$

**Lemma 24.** *The metric  $\rho'$  is the maximum of  $d'_1, \dots, d'_m$ .*

*Proof.* Let  $x, y \in \prod_{i \in I} M_i$ . For  $1 \leq k \leq m$ , let  $I_k = \{i \in I : \rho_i(x_i, y_i) = d_{k,i}(x_i, y_i)\}$ . Note that  $I = \bigcup_{k \leq m} I_k$ . Since  $D$  is an ultrafilter on  $I$ , there must exist  $j \leq m$  such that  $I_j \in D$ . For if there does not exist such a  $j$ , then for all  $1 \leq k \leq m$ , we have  $I_k^c \in D$ , and therefore  $\bigcap_{k \leq m} I_k^c \in D$ . However, since  $I = \bigcup_{k \leq m} I_k$ ,  $\bigcap_{k \leq m} I_k^c = \emptyset$ , a contradiction.

Thus, suppose  $I_j \in D$ . By definition of  $\rho_i$ , we have  $\{i \in I : d_{j,i}(x_i, y_i) \geq d_{l,i}(x_i, y_i) \text{ for all } 1 \leq l \leq m\} \in D$ . Therefore,  $\lim_{i, D} d_{j,i}(x_i, y_i) \geq \lim_{i, D} d_{l,i}(x_i, y_i)$  for all  $1 \leq l \leq m$ . Since  $I_j \in D$ , we also have  $\lim_{i, D} \rho_i(x_i, y_i) = \lim_{i, D} d_{j,i}(x_i, y_i)$ . Therefore,  $\rho'(x, y) = d'_j(x, y) = \max\{d'_1(x, y), \dots, d'_m(x, y)\}$ .  $\square$

**Lemma 25.** *The functions  $d'_1, \dots, d'_m$  are well-defined.*

*Proof.* Fix  $k$  with  $1 \leq k \leq m$ . Let  $[a], [b], [c], [d] \in \prod_{i \in I} M_i / \sim$ , and suppose  $\lim_{i, D} \rho(a, b) = 0$  and  $\lim_{i, D} \rho(c, d) = 0$ , so that  $\rho'([a], [b]) = 0$  and  $\rho'([c], [d]) = 0$ . We

want to show that  $d'_k([a], [c]) = d'_k([b], [d])$ . Since  $\rho' = \max\{d'_1, \dots, d'_m\}$ , we have

$$\begin{aligned} |d'_k([a], [c]) - d'_k([b], [d])| &= |d'_k([a], [c]) - d'_k([c], [b]) + d'_k([c], [b]) - d'_k([b], [d])| \\ &\leq |d'_k([a], [b])| + |d'_k([c], [d])| \\ &\leq |\rho'([a], [b])| + \rho'([c], [d]) \\ &= 0 \end{aligned}$$

Therefore,  $d'_k$  is well-defined.  $\square$

**Lemma 26.** *For  $1 \leq k \leq m$ ,  $d'_k$  is a metric on  $\prod_{i \in I} M_i / \sim$ .*

*Proof.* Fix  $k$  with  $1 \leq k \leq m$ . It is clear that the symmetry property for metrics holds, and the triangle inequality holds by the properties of  $D$ -limits and the fact that for each  $i \in I$ ,  $d_{k,i}$  is a metric. It is also clear that for all  $[x], [y] \in \prod_{i \in I} M_i / \sim$ ,  $d'_k([x], [y]) \geq 0$ . Now suppose  $d'_k([x], [y]) = 0$ . We want to show that  $[x] = [y]$ . If  $d'_k([x], [y]) = 0$ , then since  $d'_k$  and  $d'_j$  are related by uniform continuity for  $1 \leq j \leq m$ ,  $d'_j([x], [y]) = 0$  for all  $1 \leq j \leq m$ . Therefore, since  $\rho'$  is still the maximum of  $d'_1, \dots, d'_m$ ,  $\rho'([x], [y]) = 0$ , and since  $\rho'$  is a metric,  $[x] = [y]$ . Conversely, if  $[x] = [y]$ , then  $\rho'([x], [y]) = 0$ , so by Lemma 22,  $d'_k([x], [y]) = 0$ .  $\square$

Note that Lemmas 21-24 hold no matter which metric we use to define the equivalence relation  $\sim$ . Thus, since  $d'_j$  and  $d'_k$  are related by uniform continuity for  $1 \leq j, k \leq m$ , and since  $\rho' = \max\{d'_1, \dots, d'_m\}$ , for  $x, y \in \prod_{i \in I} M_i$  and for any  $1 \leq k \leq m$ ,  $\rho(x, y) = 0$  if and only if  $d_k(x, y) = 0$ . Therefore, it does not matter which metric we use to define the equivalence relation  $\sim$ , as we obtain the same equivalence classes and all of  $d'_1, \dots, d'_m, \rho'$  are metrics on the ultraproduct.

**Notation.** Henceforth, we will denote the ultraproduct  $\prod_{i \in I} M_i / \sim$  by  $(\prod_{i \in I} M_i)_D$ . We will denote the ultrapower of  $M$  for an ultrafilter  $D$  by  $(M)_D$  or  $\prod_D M$ .

Let  $T : M \rightarrow (M)_D$  (where  $(M)_D$  is the ultrapower of  $M$ ) be the map defined by  $T(x) = ((x_i)_{i \in I})_D$ , where  $x_i = x$  for each  $i \in I$ . Note that by our definition of the metrics  $d'_1, \dots, d'_m, \rho'$ ,  $T$  preserves each of these metrics. In other words, for  $1 \leq k \leq m$ ,  $d_k(x, y) = d'_k(T(x), T(y))$ , and  $\rho(x, y) = \rho'(T(x), T(y))$ . As we will see,  $T$  is also an embedding. It is also still the case that if  $(M, d)$  is compact, then  $T : M \rightarrow (M)_D$  is surjective. Note that we will only apply this fact when  $M = [0, 1]$  and  $d$  is the standard metric on  $[0, 1]$ .

Let  $((M_i, d_{1,i}, \dots, d_{m,i}, \rho_i) : i \in I)$  be a uniformly equicontinuous family of complete (with respect to any, and therefore all, of the metrics), bounded metric spaces, all with diameter  $\leq K$ . Since each  $M_i$  was complete for all metrics and since it does not matter which metric we use to define the equivalence relation on  $\prod_{i \in I} M_i$ , the proof of Proposition 5.3 in [2] shows that  $((\prod_{i \in I} M_i)_D, d'_1, \dots, d'_m, \rho')$  is complete for all metrics. Therefore, the ultraproduct of a uniformly equicontinuous class of multimetric structures is a multimetric structure.

We define ultraproducts of functions in exactly the same way as in [2]. Since we assign  $m + 1$  moduli of uniform continuity to each function symbol  $f$ , each ultraproduct of functions also has  $m + 1$  moduli of uniform continuity, one for each metric on the ultraproduct. The proof of this fact is the same as the proof in the single-metric case in [2].

We also define ultraproducts of predicates in the same way as in [2]. Note that when we consider ultraproducts of predicates, we have  $m + 1$  metrics on the domain,



and we can consider the range to also have  $m + 1$  metrics (which are all the same). We still obtain  $m + 1$  moduli of continuity for each ultraproduct of predicates. Since  $[0, 1]$  is compact, the ultraproduct of  $[0, 1]$  with the standard metric (or  $m + 1$  copies of the standard metric) is isomorphic to  $[0, 1]$  with the standard metric. So we consider predicates to have values in  $[0, 1]$ , by replacing the interpretation of the original predicate with a composition of functions.

In the next section, we prove Łoś's theorem for multimetric structures. The statement of the theorem is the same as in [2], but the proof is different.

### 8. ŁOŚ'S THEOREM

For this section, fix a set  $I$ , an ultrafilter  $D$  on  $I$ , and a language  $L$ . Throughout this section, let  $(\mathcal{M}_i : i \in I)$  be a uniformly equicontinuous family of  $L$ -structures, and let  $\mathcal{M} = (\prod_{i \in I} \mathcal{M}_i)_D$ . Before we prove Łoś's theorem, we must first prove a lemma.

**Lemma 27.** *Let  $t(x_1, \dots, x_n)$  be an  $L$ -term. For  $a_k = ((a_i^k)_{i \in I})_D \in M$ ,  $1 \leq k \leq n$ , we have*

$$t^{\mathcal{M}}(a_1, \dots, a_n) = ((t^{\mathcal{M}_i}(a_i^1, \dots, a_i^n))_{i \in I})_D$$

*Proof.* We prove this by induction on complexity of terms.

First suppose  $t$  is the variable  $x$ . Then  $t^{\mathcal{M}}(a) = a$ , by definition. But we have  $a = ((a_i)_{i \in I})_D$  and  $t^{\mathcal{M}_i}(a_i) = a_i$ , so the equality holds. If  $t$  is the constant symbol  $c$ , then the lemma holds by definition of  $c^{\mathcal{M}}$ .

Now suppose that  $f$  is a  $k$ -ary function and

$$t(x_1, \dots, x_n) = f(t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n))$$

where  $t_1, \dots, t_k$  satisfy the conclusion of the lemma. We have  $t^{\mathcal{M}}(a_1, \dots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_n), \dots, t_k^{\mathcal{M}}(a_1, \dots, a_n))$ . But by our induction hypothesis, for  $1 \leq j \leq k$ ,  $t_j^{\mathcal{M}}(a_1, \dots, a_n) = ((t_j^{\mathcal{M}_i}(a_i^1, \dots, a_i^n))_{i \in I})_D$ . By definition of  $f^{\mathcal{M}}$ , we have  $f^{\mathcal{M}} = (\prod_{i \in I} f^{\mathcal{M}_i})_D$ . So by definition of  $(\prod_{i \in I} f^{\mathcal{M}_i})_D$ , the conclusion of the lemma holds in this case as well.  $\square$

We are now ready to state and prove Łoś's theorem for multimetric structures.

**Theorem (Łoś).** *Let  $\varphi(x_1, \dots, x_n)$  be an  $L$ -formula. For  $a_k = ((a_i^k)_{i \in I})_D \in M$ ,  $1 \leq k \leq n$ , we have*

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i, D} \varphi^{\mathcal{M}_i}(a_i^1, \dots, a_i^n)$$

*Proof.* In order to prove this theorem, we again use induction on complexity. Throughout this proof, let  $d$  denote the standard metric on  $[0, 1]$ .

The atomic formulas are  $P(t_1, \dots, t_n)$ , where  $P$  is a predicate and  $t_i$  are terms, and  $d_1(t_1, t_2), \dots, d_m(t_1, t_2), \rho(t_1, t_2)$ . Let  $\varphi(x_1, \dots, x_k)$  be the formula  $P(t_1(x_1, \dots, x_k), \dots, t_n(x_1, \dots, x_k))$ . We want to show that  $\lim_{i, D} \varphi^{\mathcal{M}_i}(a_i^1, \dots, a_i^k) = \varphi^{\mathcal{M}}(a_1, \dots, a_k)$  (where  $a_1 = ((a_i^1)_{i \in I})_D$ , etc.). Suppose  $\varphi^{\mathcal{M}}(a_1, \dots, a_k) = r$ , where  $r \in [0, 1]$ . Then by the previous lemma, the fact that  $D$ -limits are unique in a compact metric space, and the definition of  $P^{\mathcal{M}}$ ,  $\lim_{i, D} \varphi^{\mathcal{M}_i}(a_i^1, \dots, a_i^k) = r$ . This proves the theorem for predicates.

We must now show that the conclusion holds for each of the metrics. Fix  $k$  with  $1 \leq k \leq m$  and consider  $d_k^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_n), t_2^{\mathcal{M}}(a_1, \dots, a_n))$ . By the previous lemma, this is equal to  $d_k^{\mathcal{M}}(((t_1^{\mathcal{M}_i}(a_i^1, \dots, a_i^n))_{i \in I})_D, ((t_2^{\mathcal{M}_i}(a_i^1, \dots, a_i^n))_{i \in I})_D)$ . But by definition, this is equal to  $\lim_{i,D} d_k^{\mathcal{M}_i}(t_1^{\mathcal{M}_i}(a_i^1, \dots, a_i^n), t_2^{\mathcal{M}_i}(a_i^1, \dots, a_i^n))$ , which is what we wanted. A similar proof shows the conclusion holds for the other metrics.

Now suppose  $u : [0, 1]^k \rightarrow [0, 1]$  is continuous and  $\varphi_1, \dots, \varphi_k$  are  $L$ -formulas such that the conclusion in this theorem holds for  $\varphi_1, \dots, \varphi_k$ . Let  $\psi(x_1, \dots, x_n)$  denote the formula  $u(\varphi_1(x_1, \dots, x_n), \dots, \varphi_k(x_1, \dots, x_n))$ . We want to show that  $\psi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i,D} \psi^{\mathcal{M}_i}(a_i^1, \dots, a_i^n)$ .

First note that  $\lim_{i,D} (b_i^1, \dots, b_i^n) = (\lim_{i,D} b_i^1, \dots, \lim_{i,D} b_i^n)$  for  $b_i^1, \dots, b_i^n \in [0, 1]$ . To see this, let  $\varepsilon > 0$  and let  $U = \{(y_1, \dots, y_n) \in [0, 1]^n : \widehat{d}((y_1, \dots, y_n), (\lim_{i,D} b_i^1, \dots, \lim_{i,D} b_i^n)) < \varepsilon\}$ . Here  $\widehat{d}$  is the metric on  $[0, 1]^n$  given by  $\widehat{d}(x, y) = \max\{d(x_1, y_1), \dots, d(x_n, y_n)\}$ , where  $x = (x_1, \dots, x_k)$  and  $y = (y_1, \dots, y_k)$ . We want to show that  $\{i \in I : (b_i^1, \dots, b_i^n) \in U\} \in D$ . Denote this last set by  $A$ . For  $1 \leq m \leq n$ , we have  $\{i \in I : d(b_i^m, \lim_{i,D} b_i^m) < \varepsilon\} \in D$ . Let  $S = \bigcap_{m \leq n} \{i \in I : d(b_i^m, \lim_{i,D} b_i^m) < \varepsilon\}$ . We will show that  $S = \{i \in I : (b_i^1, \dots, b_i^n) \in U\}$ . First let  $i \in S$ . Then for  $1 \leq m \leq n$ , we have  $d(b_i^m, \lim_{i,D} b_i^m) < \varepsilon$ . Therefore, by the definition of the metric  $\widehat{d}$  on  $[0, 1]^n$ , we have  $i \in A$ . Now let  $i \in A$ . Then  $\max\{d(b_i^1, \lim_{i,D} b_i^1), \dots, d(b_i^n, \lim_{i,D} b_i^n)\} < \varepsilon$  by definition of  $d$ . Then  $i \in S$  by definition of  $S$ . So  $A \in D$ , and  $\lim_{i,D} (b_i^1, \dots, b_i^n) = (\lim_{i,D} b_i^1, \dots, \lim_{i,D} b_i^n)$  when  $b_i^1, \dots, b_i^n \in [0, 1]$ . Now by our induction hypothesis that  $\varphi_j^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i,D} \varphi_j^{\mathcal{M}_i}(a_i^1, \dots, a_i^n)$  for each  $1 \leq j \leq k$  and the fact that  $u$  is continuous,  $\psi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i,D} \psi^{\mathcal{M}_i}(a_i^1, \dots, a_i^n)$  by Lemma 5.1 in [2].

Now suppose  $\varphi$  is an  $L$ -formula such that the conclusion in Łoś's theorem holds for  $\varphi$  and consider  $\sup_y \varphi$ . Let  $\sigma(x_1, \dots, x_n) = \sup_y \varphi(y, x_1, \dots, x_n)$ . We want to show that  $\sigma^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i,D} \sigma^{\mathcal{M}_i}(a_i^1, \dots, a_i^n)$ , assuming that the conclusion in Łoś's theorem holds for  $\varphi$ . In other words, we want to show that

$$\sup_{y \in M} \varphi^{\mathcal{M}}(y, a_1, \dots, a_n) = \lim_{i,D} \sup_{y_i \in M_i} \varphi^{\mathcal{M}_i}(y_i, a_i^1, \dots, a_i^n) \quad (*)$$

Let  $r_i = \sup_{y_i \in M_i} \varphi^{\mathcal{M}_i}(y_i, a_i^1, \dots, a_i^n)$ , and let  $r = \lim_{i,D} r_i$ . (Thus,  $r$  is the right side of equation (\*).) For each  $\delta > 0$ , let  $A(\delta) \in D$  be such that  $r - \delta < r_i < r + \delta$  for every  $i \in A(\delta)$ . For each  $i \in A(\delta)$  and for all  $c \in M_i$ , we have  $\varphi^{\mathcal{M}_i}(c, a_i^1, \dots, a_i^n) \leq r_i < r + \delta$ . Therefore, for  $b = ((b_i)_{i \in I})_D \in M$ , we must have  $\lim_{i,D} \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) \leq r$ . To see this, suppose  $\lim_{i,D} \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) > r$ , and denote the limit by  $L_b$ . Then there exists  $\varepsilon > 0$  such that  $\lim_{i,D} \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) > r + \varepsilon > r$ . By definition of  $D$ -limit, there exists  $\lambda > 0$  such that  $\{i \in I : d(L_b, \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n)) < \lambda\} \in D$  and  $L_b - \lambda > r + \varepsilon$ . But the complement of this set must be in  $D$ , since the complement

contains  $A(\varepsilon)$ , a contradiction. So  $L_b \leq r$ . Therefore, since  $b \in M$  was arbitrary, by our induction hypothesis, we have  $\varphi^{\mathcal{M}}(c, a_1, \dots, a_k) \leq r$  for all  $c \in M$ .

We now show that  $\sup_{y \in M} \varphi^{\mathcal{M}}(y, a_1, \dots, a_n) = r$ . Fix  $\varepsilon > 0$  and for each  $i \in I$  let  $b_i \in M_i$  be such that  $r_i \leq \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) + \varepsilon/2$ . Such a  $b_i$  exists by the definition of supremum and  $r_i$ . In particular, for all  $i \in A(\varepsilon/2)$ , we have  $r_i + \varepsilon/2 \leq \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) + \varepsilon$ . Since  $r - \varepsilon/2 < r_i$  for all  $i \in A(\varepsilon/2)$ , we now have  $r < \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) + \varepsilon$  for all  $i \in A(\varepsilon/2)$ . Now let  $b = ((b_i)_{i \in I})_D$ . By our induction hypothesis,  $\lim_{i, D} \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n) = \varphi^{\mathcal{M}}(b, a_1, \dots, a_n)$ . Therefore, in order to prove that the supremum is  $r$ , we show that  $r - \varepsilon \leq \lim_{i, D} \varphi^{\mathcal{M}_i}(b_i, a_i^1, \dots, a_i^n)$ . But this is clear, since  $A(\varepsilon/2) \in D$ . So there exists  $b \in M$  such that  $r \leq \varphi^{\mathcal{M}}(b, a_1, \dots, a_n) + \varepsilon$ , and therefore,  $\sup_{y \in M} \varphi^{\mathcal{M}}(y, a_1, \dots, a_k) = r$ .

The analogous statements for infimum are proved similarly, or by replacing each  $\varphi^{\mathcal{M}_i}$  by its negative. □

Once we have Łoś's theorem, Corollaries 5.5 and 5.6 in [2] follow immediately. Theorem 5.7 of [2] (the Keisler-Shelah theorem) still holds for multimetric structures, but we skip the proof for now. Theorem 5.8 of [2], which is a continuous model theory version of the compactness theorem, also holds for restricted classes of structures.

**Theorem 28.** *Let  $T$  be an  $L$ -theory and  $\mathcal{C}$  a uniformly equicontinuous class of  $L$ -structures. If  $T$  is finitely satisfiable in  $\mathcal{C}$ , then there exists an ultraproduct of structures from  $\mathcal{C}$  that models  $T$ .*

The proof of this theorem is exactly the same as the proof of Theorem 5.8 in [2].

## 9. CARDINALITY OF $L$ -FORMULAS

Before moving on, we make a note about the cardinality of the set of  $L$ -formulas for a given signature  $L$ .

**Definition 29.** For a signature  $L$ , let  $|L|$  denote the cardinality of the set of predicate, function, and constant symbols of  $L$ . We define the cardinality of  $L$ , denoted  $\text{card}(L)$ , to be  $\omega \cup |L|$ .

We define a system of connectives  $\mathcal{F}$  as in Definition 6.1 of [2], and define an  $\mathcal{F}$ -restricted formula as in Definition 6.2 of [2].

We would like to say that if  $\text{card}(L) \leq \kappa$  for some cardinal  $\kappa$ , then the cardinality of the set of  $L$ -formulas is also at most  $\kappa$ . However, we allow any continuous function  $u : [0, 1]^n \rightarrow [0, 1]$  for any  $n \geq 1$  as a connective, and thus the cardinality of the set of  $L$ -formulas may be greater than  $\kappa$ . As proved in Chapter 6 of [2], there exists a countable set of connectives  $\mathcal{F}_0$  such that for any  $\varepsilon > 0$  and any  $L$ -formula  $\varphi(x_1, \dots, x_n)$ , there exists an  $\mathcal{F}_0$ -restricted  $L$ -formula  $\psi(x_1, \dots, x_n)$  such that for all  $L$ -structures  $\mathcal{M}$ ,

$$|\varphi^{\mathcal{M}}(a_1, \dots, a_n) - \psi^{\mathcal{M}}(a_1, \dots, a_n)| \leq \varepsilon$$

for all  $a_1, \dots, a_n \in M$ .

Since  $\mathcal{F}_0$  is countable, if  $\text{card}(L) \leq \kappa$ , then the set of  $\mathcal{F}_0$ -restricted  $L$ -formulas has cardinality at most  $\kappa$  as well. Since we can approximate  $L$ -formulas by  $\mathcal{F}_0$ -restricted  $L$ -formulas, from here on, we will assume that if  $\text{card}(L) \leq \kappa$ , then the set of  $L$ -formulas has cardinality at most  $\kappa$ .

## 10. SATURATED MODELS

In this section, we prove that two theorems from first-order logic regarding saturated models hold in multimetric continuous model theory.

Our definition of satisfiability of a set  $\Gamma(x_1, \dots, x_n)$  of  $L$ -conditions in an  $L$ -structure  $\mathcal{M}$  is the same as in [2]. Our definition of a  $\kappa$ -saturated model ( $\kappa$  an infinite cardinal) is also the same as Definition 7.5 in [2].

If  $\{x_i\}_{i \in I}$  is a set of real numbers and  $\inf\{x_i\}_{i \in I} = 0$ , there does not necessarily exist  $i \in I$  such that  $x_i = 0$ . However, the next proposition (which is a variation of Proposition 7.7 in [2]) gives conditions under which we can replace “inf” by “there exists” and “sup” by “for all” in  $L$ -formulas.

**Proposition 30.** *Let  $\mathcal{M}$  be an  $L$ -structure and suppose  $E(x_1, \dots, x_m)$  is the  $L$ -condition  $Q_{y_1}^1 \dots Q_{y_n}^n \varphi(x_1, \dots, x_m, y_1, \dots, y_n) = 0$  where each  $Q^i$  is either inf or sup and  $\varphi$  is not necessarily quantifier free, but does not have any quantifiers in its front. Let  $\mathcal{E}(x_1, \dots, x_m)$  be the statement*

$$\tilde{Q}^1 y_1 \dots \tilde{Q}^n y_n (\varphi(x_1, \dots, x_m, y_1, \dots, y_n) = 0)$$

where each  $\tilde{Q}^i y_i$  is  $\exists y_i$  if  $Q_{y_i}^i$  is  $\inf_{y_i}$  and  $\forall y_i$  if  $Q_{y_i}^i$  is  $\sup_{y_i}$ . If  $\mathcal{M}$  is  $\omega$ -saturated, then

$$\mathcal{M} \models E[a_1, \dots, a_m] \text{ if and only if } \mathcal{E}(a_1, \dots, a_m) \text{ is true in } \mathcal{M}$$

for any  $a_1, \dots, a_m \in M$ .

*Proof.* We induct on  $n$ . If  $n = 0$ , then this is clearly true. Now suppose  $\psi(x_1, \dots, x_m, y)$  (where  $m \in \mathbb{N}$  is arbitrary) is a formula of the form

$$Q_{z_1}^1 \dots Q_{z_n}^n \theta(x_1, \dots, x_m, z_1, \dots, z_n, y)$$

where each  $Q^i$  is either sup or inf and quantifies over  $z_i$ , and  $\theta$  does not have any quantifiers in its front. Let  $\mathcal{E}(x_1, \dots, x_m)$  denote the statement

$$\tilde{Q}y \tilde{Q}^1 z_1 \dots \tilde{Q}^n z_n (\theta(x_1, \dots, x_m, z_1, \dots, z_n, y) = 0)$$

where  $\tilde{Q}y$  is  $\exists y$ ,  $\tilde{Q}^i z_i$  is  $\exists z_i$  if  $Q_{z_i}^i$  is  $\inf_{z_i}$ , and  $\tilde{Q}^i z_i$  is  $\forall z_i$  if  $Q_{z_i}^i$  is  $\sup_{z_i}$ . Similarly, let  $\mathcal{E}'(x_1, \dots, x_m)$  denote the statement

$$\tilde{P}y \tilde{Q}^1 z_1 \dots \tilde{Q}^n z_n (\theta(x_1, \dots, x_m, z_1, \dots, z_n, y) = 0)$$

where  $\tilde{P}y$  is  $\forall y$  and for  $1 \leq i \leq n$ ,  $\tilde{Q}^i z_i$  are as in the statement  $\mathcal{E}$ .

Suppose that the conclusion of the theorem is true for  $\psi$ .

Let  $a_1, \dots, a_m \in M$ . It is clear that  $\sup_y \psi^{\mathcal{M}}(a_1, \dots, a_m, y) = 0$  if and only if  $\mathcal{E}'(a_1, \dots, a_m)$  is true in  $\mathcal{M}$ . It is also clear that if  $\mathcal{E}(a_1, \dots, a_m)$  is true in  $\mathcal{M}$ , then  $\inf_y \psi(a_1, \dots, a_m, y) = 0$ . Therefore, we must show that if  $\inf_y \psi^{\mathcal{M}}(a_1, \dots, a_m, y) = 0$ , then  $\mathcal{E}(a_1, \dots, a_m)$  is true in  $\mathcal{M}$ . If  $y$  is not a free variable of  $\psi$ , then this is clearly true. Now suppose  $y$  is a free variable of  $\psi$ . By our hypothesis that the infimum over  $y \in M$  of  $\psi^{\mathcal{M}}(a_1, \dots, a_m, y)$  is 0, for each  $k \in \mathbb{N}$ , there exists

$c_k \in M$  such that  $\psi^{\mathcal{M}}(a_1, \dots, a_m, c_k) \leq 1/(k+1)$ . Thus, the collection of conditions  $\psi(a_1, \dots, a_m, y) \leq 1/(k+1)$  (where  $k$  ranges over  $\mathbb{N}$ ) is finitely satisfiable in  $\mathcal{M}$ . Since  $\mathcal{M}$  is  $\omega$ -saturated and  $\{a_1, \dots, a_m\}$  is finite, there exists  $c$  such that  $\psi(a_1, \dots, a_m, c) = 0$  in  $\mathcal{M}$ . By our induction hypothesis, the statement

$$\tilde{Q}^1 z_1 \dots \tilde{Q}^n z_n \theta(a_1, \dots, a_m, c)$$

is true in  $\mathcal{M}$ . Therefore,  $\mathcal{E}(a_1, \dots, a_m)$  is true in  $\mathcal{M}$ , as desired.  $\square$

We now prove that two theorems about saturated models still hold in multimetric continuous model theory. The first theorem is the multimetric continuous model theory analogue of Theorem 5.1.11 in [4], and the proof given here is mostly similar to the proof of that theorem.

**Theorem 31.** *Suppose  $\alpha \geq \omega$ ,  $\mathcal{M}, \mathcal{N}$  are  $\alpha$ -saturated, and  $\mathcal{M} \equiv \mathcal{N}$ . Let  $a \in {}^\alpha M$  and  $b \in {}^\alpha N$  be sequences in  $M$  and  $N$  respectively, indexed by  $\alpha$ . Then there exist  $\bar{a} \in {}^\alpha M$  and  $\bar{b} \in {}^\alpha N$  such that:*

- (1)  $\text{range}(a) \subset \text{range}(\bar{a})$
- (2)  $\text{range}(b) \subset \text{range}(\bar{b})$
- (3)  $(\mathcal{M}, \bar{a}_\xi)_{\xi < \alpha} \equiv (\mathcal{N}, \bar{b}_\xi)_{\xi < \alpha}$

*Proof.* As in Theorem 5.1.11 of [4], we represent each ordinal  $\xi < \alpha$  uniquely as  $\xi = \lambda + n$ , where  $\lambda$  is a limit ordinal and  $n \in \omega$ . We call  $\xi$  even if  $n$  is even, and otherwise call  $\xi$  odd. We want to find sequences  $\bar{a} \in {}^\alpha M$ ,  $\bar{b} \in {}^\alpha N$  such that for all  $\xi < \alpha$ :

- (a) For  $\eta < \xi$ , if  $\eta = \lambda + 2n$  is even, then  $\bar{a}_\eta = a_{\lambda+n}$
- (b) For  $\eta < \xi$ , if  $\eta = \lambda + (2n+1)$  is odd, then  $\bar{b}_\eta = b_{\lambda+n}$
- (c)  $(\mathcal{M}, \bar{a}_\eta)_{\eta < \xi} \equiv (\mathcal{N}, \bar{b}_\eta)_{\eta < \xi}$

We do this by transfinite recursion. Let  $\xi < \alpha$  and suppose we have  $\bar{a}_\eta, \bar{b}_\eta$  for each  $\eta < \xi$  such that (a), (b), and (c) hold for  $\xi$ . Now suppose  $\xi = \lambda + 2n$  is even. Let  $\bar{a}_\xi = a_{\lambda+n}$ . Let  $A = \{\bar{a}_\eta\}_{\eta < \xi}$  and let  $\Sigma(x)$  be a collection of  $L(A)$ -conditions such that  $\Sigma(x)$  is the type of  $\bar{a}_\xi$  in  $\mathcal{M}_A$ . Let  $\{\varphi_1(x), \dots, \varphi_n(x)\}$  be a finite collection of formulas from  $\Sigma(x)$ , and consider the condition  $\inf_x \max(\varphi_1(x), \dots, \varphi_n(x)) = 0$ . By assumption, this holds in  $(\mathcal{M}, \bar{a}_\eta)_{\eta < \xi}$ , and so by condition (c), the corresponding formula holds in  $(\mathcal{N}, \bar{b}_\eta)_{\eta < \xi}$  as well. Since  $\mathcal{N}$  is  $\alpha$ -saturated and  $\xi < \alpha$ , by the previous theorem, there exists  $c \in N$  such that  $\max(\varphi'_1(c), \dots, \varphi'_n(c)) = 0$  in  $\mathcal{N}_B$  (where  $\varphi'_i$  is the  $L(B)$ -formula corresponding to  $\varphi_i$ ). So  $\Sigma(x)$  is finitely satisfiable in  $(\mathcal{N}, \bar{b}_\eta)_{\eta < \xi}$ , and since  $\mathcal{N}$  is  $\alpha$ -saturated, there exists  $\bar{b}_\xi \in N$  realizing  $\Sigma(x)$ . Now  $(\mathcal{M}, \bar{a}_\eta)_{\eta \leq \xi} \equiv (\mathcal{N}, \bar{b}_\eta)_{\eta \leq \xi}$ , since  $\Sigma(x)$  was the type of  $\bar{a}_\xi$ .

If  $\xi = \lambda + (2n+1)$  is odd, then let  $\bar{b}_\xi = b_{\lambda+n}$  and find  $\bar{a}_\xi$  in a similar manner as above. This defines  $\bar{a}$  and  $\bar{b}$  by transfinite recursion, and  $\bar{a} \in {}^\alpha M$  and  $\bar{b} \in {}^\alpha N$  satisfy the conclusions of the lemma.  $\square$

The next theorem is the multimetric continuous model theory analogue of Theorem 5.1.13 in [4]. Again, the proof is mostly similar to the proof of Theorem 5.1.13. This theorem gives conditions under which elementarily equivalent models are isomorphic.

**Theorem 32.** *Suppose  $\mathcal{M}, \mathcal{N}$  are  $\alpha$ -saturated  $L$ -structures,  $|M| = |N| = \alpha$ , and  $\mathcal{M} \equiv \mathcal{N}$ . Then  $\mathcal{M} \cong \mathcal{N}$ .*

*Proof.* Let  $|M| = |N| = \alpha$  and let  $a \in {}^\alpha M$ ,  $b \in {}^\alpha N$  be enumerations of  $M$  and  $N$  respectively. By Theorem 31 above, there are "extensions"  $\bar{a}$ ,  $\bar{b}$  of  $a, b$  such that  $(\mathcal{M}, \bar{a}_\xi)_{\xi < \alpha} \equiv (\mathcal{N}, \bar{b}_\xi)_{\xi < \alpha}$ . Let  $d$  be one of the metric symbols of  $L$ . Now note that if  $d^{\mathcal{M}}(\bar{a}_\eta, \bar{a}_\gamma) = 0$ , then  $d^{\mathcal{N}}(\bar{b}_\eta, \bar{b}_\gamma) = 0$ , since  $(\mathcal{M}, \bar{a}_\xi)_{\xi < \alpha} \equiv (\mathcal{N}, \bar{b}_\xi)_{\xi < \alpha}$  by Theorem 31. So the map  $\varphi : M \rightarrow N$  defined by  $\varphi(\bar{a}_\eta) = \bar{b}_\eta$  is well-defined. Similarly, if  $\bar{a}_\eta \neq \bar{a}_\gamma$ , then  $d(\bar{a}_\eta, \bar{a}_\gamma) = r_{\eta, \gamma}$  for some  $r_{\eta, \gamma} > 0$ , and so  $\bar{b}_\eta \neq \bar{b}_\gamma$ . So  $\varphi$  is also injective. Also, since  $a, b$  enumerate  $M, N$ , by conditions (1) and (2) of Theorem 31, so do  $\bar{a}, \bar{b}$ . Therefore,  $\varphi$  is a bijection between  $M$  and  $N$ .

Therefore, by Theorem 31,  $\varphi$  is an isometry (for each metric) and it preserves interpretations of functions, constants, and predicates. We check that  $\varphi$  preserves interpretations of functions. Let  $a_1, \dots, a_n \in M$  and suppose  $f^{\mathcal{M}}(a_1, \dots, a_n) = a$ . Let  $d$  be any of the metric symbols of  $L$ . Then  $d^{\mathcal{M}}(f^{\mathcal{M}}(a_1, \dots, a_n), a) = 0$ . So since  $(\mathcal{M}, \bar{a}_\xi)_{\xi < \alpha} \equiv (\mathcal{N}, \bar{b}_\xi)_{\xi < \alpha}$ , we have  $d^{\mathcal{N}}(f^{\mathcal{N}}(\varphi(a_1), \dots, \varphi(a_n)), \varphi(a)) = 0$ . This shows that  $\varphi$  preserves interpretations of functions, since  $d$  is a metric. The proofs that  $T$  preserves interpretations of constants and predicates are similar.  $\square$

## 11. AXIOMATIZABILITY

Next, we give several characterizations of axiomatizability of classes of multimetric  $L$ -structures for certain classes of  $L$ -structures. Our first theorem in this section is a multimetric continuous model theory analogue of Theorem 4.1.12(i) in [4] for uniformly equicontinuous classes of  $L$ -structures.

**Theorem 33.** *Let  $K$  be a uniformly equicontinuous class of multimetric  $L$ -structures. Then  $K$  is axiomatizable if and only if  $K$  is closed under ultraproducts and elementary equivalence.*

*Proof.* First suppose  $K$  is exactly the set of models of  $T$ , where  $T$  is a set of closed  $L$ -conditions. Then if  $\mathcal{M} \in K$  and  $\mathcal{M} \equiv \mathcal{N}$ , then  $\mathcal{N} \models T$ , so  $\mathcal{N} \in K$ . By Łoś's theorem, if  $\mathcal{M}_i \models \varphi = 0$  for all  $i \in I$ , then  $\prod_D \mathcal{M}_i \models \varphi = 0$  for any ultrafilter  $D$  over  $I$ . Therefore,  $K$  is also closed under ultraproducts.

Now suppose  $K$  is closed under ultraproducts and elementary equivalence. Let  $T$  be the set of closed  $L$ -conditions that hold in all  $\mathcal{M} \in K$ . Let  $\mathcal{N} \models T$ , and let  $\Sigma = \text{Th}(\mathcal{N})^+$ . Then  $\Sigma$  is finitely satisfiable in  $K$  (see the proof of Theorem 5.14 in [2]). By the compactness theorem for uniformly equicontinuous classes of  $L$ -structures, there exists an ultraproduct  $\mathcal{N}'$  of structures from  $K$  such that  $\mathcal{N}' \models \Sigma$ . By assumption,  $\mathcal{N}' \in K$ . Then  $\mathcal{N}' \equiv \mathcal{N}$ , so since  $K$  is closed under elementary equivalence,  $\mathcal{N} \in K$ . Therefore,  $K$  is axiomatizable.  $\square$

Next, we prove the multimetric continuous model theory analogue of the Keisler-Shelah theorem. In order to prove this, we must first give the definition of  $\kappa$ -consistent. This definition is the same as in section 6.1 of [4].

**Definition 34.** Let  $\lambda, \kappa$  be infinite cardinals and let  $\mu$  be the least cardinal  $\alpha$  such that  $\lambda^\alpha > \lambda$ . Note that  $\mu \leq \lambda$ . Let  $F$  be a set of functions  $f : \lambda \rightarrow \mu$  and let  $G$  be a set of functions  $g : \lambda \rightarrow \beta(g)$ , where  $\beta(g)$  is a cardinal less than  $\mu$ . Let  $D$  be a filter over  $\mu$ . We say that  $(F, G, D)$  is  $\kappa$ -consistent if and only if:

(i)  $D$  is generated by a subset  $E$  of power at most  $\kappa$ . This means that  $E \subset D$ ,  $|E| \leq \kappa$ ,  $E$  is closed under finite intersections, and every element of  $D$  is a superset of some element in  $E$ .

(ii) For any cardinal  $\beta < \mu$ , a sequence  $\{f_\rho\}_{\rho < \beta}$  of distinct functions in  $F$ , a sequence  $\{\sigma_\rho\}_{\rho < \beta}$  of ordinals less than  $\mu$ , and two functions  $f \in F$  and  $g \in G$ , the set

$$\{\xi < \lambda : f_\rho(\xi) = \sigma_\rho \text{ for all } \rho < \beta \text{ and } f(\xi) = g(\xi)\}$$

together with  $D$  generates a nontrivial filter over  $\lambda$ .

Here we take the trivial filter over  $\lambda$  to mean  $S(\lambda)$ , the power set of  $\lambda$ . Note that part (ii) of the definition of  $\kappa$ -consistent implies that  $D$  is a nontrivial filter over  $\lambda$ .

The next lemma gives a multimetric continuous model theory analogue of Lemma 6.1.14 in [4] (or Theorem 10.6 in [6]). The proof draws on the proof of Lemma 6.1.14 in [4] and the proof of Lemma 5 of [12].

**Lemma 35.** *Let  $\mathcal{M}$  be a metric  $L$ -structure with  $|M| < \mu$  and suppose  $(F, \emptyset, D)$  is  $\kappa$ -consistent. Let  $\varphi_\xi$ ,  $\xi < \kappa$ , be  $L$ -formulas. We assume that the set  $P = \{\varphi_\xi : \xi < \kappa\}$  is closed under the binary connective  $\max$ . We also assume that  $\varphi_\xi = \varphi_\xi(x, y_{\xi,1}, \dots, y_{\xi,n(\xi)})$ . For  $\xi < \kappa$  and  $1 \leq m \leq n(\xi)$ , let  $a_{\xi,m} : \lambda \rightarrow M$  be a function mapping  $\lambda$  into  $M$ . Suppose that for each  $\xi < \kappa$ ,*

$$A_\xi := \{\nu < \lambda : \inf_x \varphi_\xi^{\mathcal{M}}(x, a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0\} \in D$$

*Then there exist  $a : \lambda \rightarrow M$ ,  $F' \subset F$ ,  $D' \supset D$ , such that  $|F \setminus F'| \leq \kappa$ ,  $(F', \emptyset, D')$  is  $\kappa$ -consistent, and for every  $\xi < \kappa$  and all  $k \geq 1$ ,*

$$\{\nu < \lambda : \psi_{k,\xi}^{\mathcal{M}}(a(\nu), a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0\} \in D'$$

*where  $\psi_{k,\xi}(x, y_{\xi,1}, \dots, y_{\xi,n(\xi)})$  is the formula  $\varphi_\xi(x, y_{\xi,1}, \dots, y_{\xi,n(\xi)}) \div 1/k$ .*

*Proof.* Let  $|M| = \alpha$  and let  $\{c_i : i < \alpha\}$  be an enumeration of  $M$ . For  $\xi < \kappa$  and  $k \geq 1$ , we define a function  $g_{k,\xi} : \lambda \rightarrow \alpha$  as follows. For  $\nu < \lambda$ , if there exists  $c_\gamma \in M$  such that  $\psi_{k,\xi}^{\mathcal{M}}(c_\gamma, a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0$ , then let  $g_{k,\xi}(\nu) = \eta$ , where  $\eta$  is the first ordinal such that  $\psi_{k,\xi}^{\mathcal{M}}(c_\eta, a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0$ . Otherwise, if there does not exist such a  $c_\gamma$ , let  $g_{k,\xi}(\nu) = 0$ .

Let  $G = \{g_{k,\xi} : \xi < \kappa, k \geq 1\}$  and note that  $\mu + |G| \leq \kappa$ . By Lemma 6.1.12 in [4], there exists  $\bar{F} \subset F$  such that  $|F \setminus \bar{F}| \leq \kappa$  and  $(\bar{F}, G, D)$  is  $\kappa$ -consistent. Let  $f$  be any element of  $\bar{F}$ . Define  $a : \lambda \rightarrow M$  by

$$a(\nu) = \begin{cases} c_{f(\nu)} & \text{if } f(\nu) < \alpha \\ c_0 & \text{otherwise} \end{cases}$$

For  $\xi < \kappa$  and  $k \geq 1$ , let

$$B_{k,\xi} = \{\nu < \lambda : \psi_{k,\xi}^{\mathcal{M}}(a(\nu), a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0\}$$

Now let  $k_1, k_2 \geq 1$  and let  $\xi, \zeta < \kappa$ . Note that since  $P$  is closed under the binary connective  $\max$ , we have

$$\max\{\varphi_\xi(x, y_{\xi,1}, \dots, y_{\xi,n(\xi)}), \varphi_\zeta(x, y_{\zeta,1}, \dots, y_{\zeta,n(\zeta)})\} = \varphi_\gamma(x, y_{\gamma,1}, \dots, y_{\gamma,n(\gamma)})$$

for some  $\gamma < \kappa$ . Therefore, letting  $k = \min\{k_1, k_2\}$ , we have  $B_{k_1,\xi} \cap B_{k_2,\zeta} = B_{k,\gamma}$ . So the set  $\{B_{k,\xi} : k \geq 1, \xi < \kappa\}$  is closed under finite intersections.

For  $F \subset S(\lambda)$ , let  $H = \{F' \subset \lambda : J \subset F' \text{ for some } J \in F\}$ . If  $F$  is closed under finite intersections, then  $H$  is the filter generated by  $F$ , as in this case,  $H$  itself is

a filter. Now let  $D'$  be the filter generated by  $D$  and  $\{B_{k,\xi} : \xi < \kappa, k \geq 1\}$ . Let  $X = E \cup \{B_{k,\xi} : \xi < \kappa, k \geq 1\}$ , where  $E$  generates  $D$  (in the sense of part (i) of the definition of  $\kappa$ -consistent). Let  $\overline{X}$  be the closure of  $X$  under finite intersections. Note that  $|\overline{X}| \leq \kappa$  and  $\overline{X}$  generates  $D'$  in the sense of part (i) of the definition of  $\kappa$ -consistent. To see this, let  $Y \in D'$ . If  $Y = \lambda$ , then clearly there exists  $\overline{Y} \in \overline{X}$  such that  $\overline{Y} \subset Y$ . Otherwise, there exist  $Y_1, \dots, Y_n \in D \cup \{B_{k,\xi} : \xi < \kappa, k \geq 1\}$  such that  $Y_1 \cap \dots \cap Y_n \subset Y$  (see Proposition 4.1.1 in [4]). Since every element of  $D$  is a superset of some element in  $E$ , there exist  $Y'_1, \dots, Y'_n \in E \cup \{B_{k,\xi} : \xi < \kappa, k \geq 1\}$  such that  $Y'_1 \cap \dots \cap Y'_n \subset Y$ . Since  $Y'_1 \cap \dots \cap Y'_n \in \overline{X}$ , we are done.

Now let  $F' = \overline{F} \setminus \{f\}$ . We certainly have  $F' \subset F$ ,  $D' \supset D$ , and for every  $\xi < \kappa$  and  $k \geq 1$ , we have  $\{\nu < \lambda : \psi_{k,\xi}^{\mathcal{M}}(a(\nu), a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0\} \in D'$  by construction. Therefore, we only need to show that  $(F', \emptyset, D')$  is  $\kappa$ -consistent.

Suppose that  $(F', \emptyset, D')$  is not  $\kappa$ -consistent. Then there exists some cardinal  $\tau < \mu$ , a sequence of functions  $f_i$  ( $i < \tau$ ) in  $F'$ , and a sequence of ordinals  $\sigma_i$  ( $i < \tau$ ) less than  $\mu$  such that

$$A := \{\nu < \lambda : f_i(\nu) = \sigma_i \text{ for all } i < \tau\} \subset \lambda \setminus X'$$

for some  $X' \in D'$ . (Otherwise, for all sequences  $f_i$  in  $F'$ , all sequences of ordinals  $\sigma_i$  less than  $\mu$ , and all  $X' \in D'$ , there exists  $\nu < \lambda$  such that  $f_i(\nu) = \sigma_i$  for all  $i < \tau$  and  $\nu \notin \lambda \setminus X'$ , i.e.  $\nu \in X'$ . Since  $D'$  was chosen to satisfy part (i) of the definition of  $\kappa$ -consistent, this would say that  $(F', \emptyset, D')$  is  $\kappa$ -consistent, a contradiction.) Since  $\{B_{k,\xi} : k \geq 1, \xi < \kappa\}$  is closed under finite intersections, there exist  $X \in D$ ,  $\xi < \kappa$ , and  $k \geq 1$  such that  $X' \supset X \cap B_{k,\xi}$ . Thus,  $A \cap B_{k,\xi} \subset (\lambda \setminus X)$ . Therefore,  $\{\nu < \lambda : f_i(\nu) = \sigma_i \text{ for all } i < \tau \text{ and } \psi_{k,\xi}^{\mathcal{M}}(a(\nu), a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0\} \subset (\lambda \setminus X)$ .

Thus, we have

$$\{\nu < \lambda : f_i(\nu) = \sigma_i \text{ for all } i < \tau \text{ and } f(\nu) = g_{k,\xi}(\nu)\} \subset (\lambda \setminus X) \cup (\lambda \setminus A_\xi) \quad (*)$$

To see this, let  $\nu$  be an element in the set on the left side of  $(*)$ . Then  $f_i(\nu) = \sigma_i$  for all  $i < \tau$  and  $f(\nu) = g_{k,\xi}(\nu)$ . If in addition there exists  $\eta < \alpha$  such that  $\psi_{k,\xi}^{\mathcal{M}}(c_\eta, a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) = 0$ , then by definition of  $g_{k,\xi}$  and  $B_{k,\xi}$ , we have  $\nu \in (\lambda \setminus X)$ . Otherwise, if there does not exist such an  $\eta$ , then  $\inf_x \varphi_\xi^{\mathcal{M}}(x, a_{\xi,1}(\nu), \dots, a_{\xi,n(\xi)}(\nu)) \geq 1/k$ , so  $\nu \in (\lambda \setminus A_\xi)$ .

Since  $(\lambda \setminus X) \cup (\lambda \setminus A_\xi) = (\lambda \setminus (X \cap A_\xi))$  and  $X \cap A_\xi \in D$  (since  $X \in D$  and, by assumption,  $A_\xi \in D$ ), this contradicts the  $\kappa$ -consistency of  $(\overline{F}, G, D)$ .  $\square$

The next theorem gives a proof of the Keisler-Shelah theorem for continuous model theory. The proof of this theorem is similar to the proof of Theorem 10.7 in [6].

**Theorem 36.** *Let  $\mathcal{M}, \mathcal{N}$  be  $L$ -structures. Then  $\mathcal{M} \equiv \mathcal{N}$  if and only if  $\mathcal{M}$  and  $\mathcal{N}$  have isomorphic ultrapowers.*

*Proof.* If  $\prod_D \mathcal{M} \cong \prod_D \mathcal{N}$  for some ultrafilter  $D$  over a set  $I$ , then it is clear that  $\mathcal{M} \equiv \mathcal{N}$ , since any model is elementarily equivalent to any of its ultrapowers by Łoś's theorem.

Now suppose  $\mathcal{M} \equiv \mathcal{N}$ . Let  $\kappa$  be an infinite cardinal such that  $|M| \leq \kappa$ ,  $|N| \leq \kappa$ , and  $\text{card}(L) \leq 2^\kappa$ . Let  $\lambda = 2^\kappa$  and let  $\mu$  be the least cardinal  $\alpha$  such that  $\lambda^\alpha > \lambda$ . Note that  $\kappa < \mu \leq \lambda$ . Let  $(S_j : j < 2^\lambda)$  be an enumeration of the subsets of  $\lambda$ .



We now use transfinite induction to construct an ultrafilter  $D$  on  $\lambda$  and an isomorphism between  $\prod_D \mathcal{M}$  and  $\prod_D \mathcal{N}$ . By Lemma 6.1.10 in [4], there is a family  $F$  of  $2^\lambda$  functions from  $\lambda$  to  $\mu$  such that  $(F, \emptyset, \{\lambda\})$  is  $\mu$ -consistent. By Lemma 6.1.11(i) in [4],  $(F, \emptyset, \{\lambda\})$  is also  $\lambda$ -consistent. Let  $F_0 = F$ ,  $D_0 = \{\lambda\}$ . Then  $(F_0, \emptyset, D_0)$  is  $\lambda$ -consistent. We now construct a decreasing sequence of sets of functions  $F_\rho$  ( $\rho < 2^\lambda$ ), an increasing sequence of proper filters  $D_\rho$  ( $\rho < 2^\lambda$ ) on  $\lambda$ , and sequences  $a_\rho : \lambda \rightarrow M$  and  $b_\rho : \lambda \rightarrow N$  such that the sequences  $a_\rho$  and  $b_\rho$  exhaust the elements of  $M^\lambda$  and  $N^\lambda$  respectively (note that  $|M^\lambda|, |N^\lambda| \leq 2^\lambda$ ) and the following conditions hold for all  $\rho < 2^\lambda$ :

- (1) (a)  $|F_0 \setminus F_\rho| \leq \lambda + |\rho|$  (this implies that  $|F_\rho| = 2^\lambda$ , since  $|F_0| = 2^\lambda$ )  
 (b)  $(F_\rho, \emptyset, D_\rho)$  is  $\lambda + |\rho|$ -consistent
- (2) If  $\rho = j + 1$  ( $j$  an ordinal), then either  $S_j \in D_\rho$  or  $\lambda \setminus S_j \in D_\rho$
- (3) If  $\rho = j + 1$  and  $j$  is even (i.e.  $j = \beta + n$  for some limit ordinal  $\beta$  and even natural number  $n$ ), then there exists a filter  $D'_j$  on  $\lambda$  such that  $D_j \subseteq D'_j \subseteq D_\rho$  and for any  $L$ -formula  $\varphi(x, x_1, \dots, x_n)$  and any  $j_1, \dots, j_n < j$ :
  - (a) Either  $\{\nu < \lambda : \varphi^{\mathcal{M}}(a_j(\nu), a_{j_1}(\nu), \dots, a_{j_n}(\nu)) = 0\}$  or its complement is in  $D'_j$
  - (b) For all  $k \geq 1$ ,  $\{\nu < \lambda : \varphi^{\mathcal{M}}(a_j(\nu), a_{j_1}(\nu), \dots, a_{j_n}(\nu)) = 0\} \in D'_j$  implies  $\{\nu < \lambda : \psi_k^{\mathcal{N}}(b_j(\nu), b_{j_1}(\nu), \dots, b_{j_n}(\nu)) = 0\} \in D_\rho$ , where  $\psi_k = \varphi \div 1/k$
- (4) If  $\rho = j + 1$  and  $j$  is odd, then there exists a filter  $D'_j$  on  $\lambda$  such that  $D_j \subseteq D'_j \subseteq D_\rho$  and for every  $L$ -formula  $\varphi(x, x_1, \dots, x_n)$  and any  $j_1, \dots, j_n < j$ :
  - (a) Either the set  $\{\nu < \lambda : \varphi^{\mathcal{N}}(b_j(\nu), b_{j_1}(\nu), \dots, b_{j_n}(\nu)) = 0\}$  or its complement is in  $D'_j$
  - (b) For all  $k \geq 1$ ,  $\{\nu < \lambda : \varphi^{\mathcal{N}}(b_j(\nu), b_{j_1}(\nu), \dots, b_{j_n}(\nu)) = 0\} \in D'_j$  implies  $\{\nu < \lambda : \psi_k^{\mathcal{M}}(a_j(\nu), a_{j_1}(\nu), \dots, a_{j_n}(\nu)) = 0\} \in D_\rho$
- (5) If  $\eta$  is a limit ordinal, then

$$F_\eta = \bigcap_{\rho < \eta} F_\rho, \quad D_\eta = \bigcup_{\rho < \eta} D_\rho$$

Note that for every subset  $S$  of  $\lambda$ , either  $S$  or its complement is in some  $D_\rho$  for some  $\rho < 2^\lambda$ , so  $D = \bigcup_{\rho < 2^\lambda} D_\rho$  is our desired ultrafilter. The isomorphism  $T : \prod_D \mathcal{M} \rightarrow \prod_D \mathcal{N}$  is defined by  $T(((a_\rho(i))_{i \in \lambda})_D) = ((b_\rho(i))_{i \in \lambda})_D$  for  $\rho < 2^\lambda$ . By the way we construct the sequences  $a_\rho$  and  $b_\rho$  ( $\rho < 2^\lambda$ ),  $T$  is surjective. Now suppose conditions (1)-(5) above hold. Let  $\varphi(x_1, \dots, x_n)$  be an  $L$ -formula and suppose  $\varphi^{\prod_D \mathcal{M}}(a_1, \dots, a_n) = 0$  for some  $a_1, \dots, a_n \in \prod_D \mathcal{M}$  with  $a_k = ((a_{\rho_k}(i))_{i \in \lambda})_D$  for some  $\rho_k < 2^\lambda$ . Then for  $k \geq 1$ , we have  $\{i < \lambda : \varphi^{\mathcal{M}}(a_{\rho_1}(i), \dots, a_{\rho_n}(i)) \leq 1/k\} \in D$  by Łoś's theorem. Let  $j$  be the least even ordinal such that  $\rho_1, \dots, \rho_n < j$ . Then by 3(a), for  $k \geq 1$ , we have  $\{i < \lambda : \varphi^{\mathcal{M}}(a_{\rho_1}(i), \dots, a_{\rho_n}(i)) \leq 1/k\} \in D'_j$ . Therefore, by 3(b), for  $k \geq 1$ , we have  $\{i < \lambda : \varphi^{\mathcal{N}}(b_{\rho_1}(i), \dots, b_{\rho_n}(i)) \leq 1/k\} \in D_{j+1} \subseteq D$ . Using Łoś's theorem again, this means that  $\varphi^{\prod_D \mathcal{N}}(b_1, \dots, b_n) = 0$ , where  $b_k = ((b_{\rho_k}(i))_{i \in \lambda})_D$ . So  $\prod_D \mathcal{M} \cong \prod_D \mathcal{N}$ .

Now suppose  $\eta$  is a limit ordinal, and suppose (1)-(5) above hold for all  $\rho < \eta$ . Define  $F_\eta$  and  $D_\eta$  as in (5) above. By Remark 10.2(2) in [6] (or Lemma 6.1.11(ii) in [4]),  $(F_\eta, \emptyset, D_\eta)$  is  $\lambda + |\eta|$ -consistent. Also, since  $F_0 \setminus F_\eta = \bigcup_{\rho < \eta} (F_0 \setminus F_\rho)$  and  $|F_0 \setminus F_\rho| \leq \lambda + |\rho|$  by assumption,  $|F_0 \setminus F_\eta| \leq \lambda + |\eta|$ . This shows that (1) holds for  $(F_\eta, \emptyset, D_\eta)$ . Since (2)-(4) only apply when  $\rho$  is a successor ordinal, we are done with this case.

Therefore, suppose  $i = j + 1$ ,  $j$  an ordinal, and suppose that (1)-(5) hold for all  $\rho \leq j$ . We assume that  $j$  is even, since the construction will be similar if  $j$  is odd. Let  $a_j$  be the first element of  $M^\lambda$  that does not appear in the list  $a_\rho$ ,  $\rho < j$ . For  $\varphi(x, y_1, \dots, y_n)$  an  $L$ -formula and  $j_1, \dots, j_n < j$  ordinals, let

$$X(\varphi, j_1, \dots, j_n) = \{\nu < \lambda : \varphi^{\mathcal{M}}(a_j(\nu), a_{j_1}(\nu), \dots, a_{j_n}(\nu)) = 0\}$$

There are at most  $\lambda + |i|$  such sets (since we assume that the set of  $L$ -formulas has cardinality at most  $\lambda$ ). Since  $(F_j, \emptyset, D_j)$  is  $\lambda + |j|$ -consistent (by our induction hypothesis), it is  $\lambda + |i|$ -consistent by Lemma 6.1.11 in [4]. Moreover, by Lemma 10.5 in [6] (or Lemma 6.1.13(ii) in [4]), there exist  $F'_j \subseteq F_j$ ,  $D'_j \supseteq D_j$  such that  $|F_j \setminus F'_j| \leq \lambda + |i|$ ,  $(F'_j, \emptyset, D'_j)$  is  $\lambda + |i|$ -consistent, for every formula  $\varphi(x, y_1, \dots, y_n)$  and ordinals  $j_1, \dots, j_n < j$ , either  $X(\varphi, j_1, \dots, j_n)$  or its complement is in  $D'_j$ , and either  $S_j$  or its complement is in  $D'_j$ .

Now fix a formula  $\varphi(x, y_1, \dots, y_n)$  and  $j_1, \dots, j_n < j$ . Suppose  $X(\varphi, j_1, \dots, j_n) \in D'_j$ . Let

$$Y(\varphi, j_1, \dots, j_n) = \{\nu < \lambda : \inf_x \varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) = 0\}.$$

Since  $X(\varphi, j_1, \dots, j_n) \subseteq Y(\varphi, j_1, \dots, j_n)$ ,  $Y(\varphi, j_1, \dots, j_n) \in D'_j$ . Again, there are at most  $\lambda + |i|$  such sets  $Y$ . Let  $k \geq 1$  and let

$$Z(\psi_k, j_1, \dots, j_n) = \{\nu < \lambda : \inf_x \psi_k^{\mathcal{N}}(x, b_{j_1}(\nu), \dots, b_{j_n}(\nu)) = 0\}$$

We claim that  $Z(\psi_k, j_1, \dots, j_n) \in D'_j$  for all  $k \geq 1$ . For suppose  $Z(\psi_k, j_1, \dots, j_n) \notin D'_j$  for some  $k \geq 1$ . Let  $l = \max\{j_1, \dots, j_n\}$ . First suppose  $l$  is odd. Since  $D'_j \supseteq D_j \supseteq D'_l$ , we must have  $Z(\psi_k, j_1, \dots, j_n) \notin D'_l$ . Applying condition 4(a) to  $i = l + 1$ , we must have  $\lambda \setminus Z(\psi_k, j_1, \dots, j_n) \in D'_l$ , so  $\lambda \setminus Z(\psi_k, j_1, \dots, j_n) \in D'_l$ . Now suppose  $l$  is even. Then since  $D'_j \supseteq D_j \supseteq D'_{l+1}$  (since  $l$  is even and  $j$  is even, we have  $l + 1 < j$ ), we have  $Z(\psi_k, j_1, \dots, j_n) \notin D'_{l+1}$ . Thus, we apply 4(a) to  $i = (l + 1) + 1$ : since  $l$  is even, we must have  $(l + 1) + 1 \leq j$ , so we can apply our induction hypothesis. Since the free variables of  $\varphi$  are among  $x, x_1, \dots, x_n$  and  $j_1, \dots, j_n < l + 1$ , we have  $\lambda \setminus Z(\psi_k, j_1, \dots, j_n) \in D'_{l+1}$ .

Let  $C = \{\nu < \lambda : \inf_x \psi_k^{\mathcal{N}}(x, b_{j_1}(\nu), \dots, b_{j_n}(\nu)) > 0\}$ . Since  $\lambda \setminus Z(\psi_k, j_1, \dots, j_n) \in D'_l$  (or  $D'_{l+1}$ ), we have  $C \in D'_l$  (or  $D'_{l+1}$ ). Since  $C$  is a subset of  $\{\nu < \lambda : \sup_x (1/k \div \varphi^{\mathcal{N}}(x, b_{j_1}(\nu), \dots, b_{j_n}(\nu))) = 0\}$ , this latter set is in  $D'_l$  (or  $D'_{l+1}$ ) as well. By condition 4(b) (applied to  $i = l + 1$  or  $i = (l + 1) + 1$ , depending on whether  $l$  is even or odd), for any  $k' > k$ ,

$$\{\nu < \lambda : \sup_x (1/k \div \varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu))) \div 1/k' = 0\} \in D_j \subseteq D'_j \quad (*)$$

For suppose  $\nu$  is in the set on the left side of  $(*)$ . Then for all  $x \in M$ , we have  $1/k \div (\varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) + 1/k') = 0$ . (See the remark following Definition 6.4 in [2].) In other words, for all  $x \in M$ ,  $\varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) + 1/k' \geq 1/k$ , or  $\varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) \geq (1/k - 1/k') \geq 1/m$  for some  $m \geq 1$ . Thus,  $\inf_x \varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) > 0$ . Let  $E = \{\nu < \lambda : \inf_x \varphi^{\mathcal{M}}(x, a_{j_1}(\nu), \dots, a_{j_n}(\nu)) > 0\}$ . By what we have just shown,  $E \in D'_j$ , so  $E \cap Y(\varphi, j_1, \dots, j_n) = \emptyset \in D'_j$ , contradicting the fact that  $D'_j$  is a proper filter. Therefore,  $Z(\psi_k, j_1, \dots, j_n) \in D'_j$  for all  $k \geq 1$ .

Note that in the case  $j = 0$ , there do not exist  $j_1, \dots, j_n < j$ . In this case, we define  $X$  and  $Y$  as above and get  $F'_0$  and  $D'_0$  by applying Lemma 10.5 in [6]. Suppose  $X(\varphi) \in D'_0$ . Then as before,  $Y(\varphi) \in D'_0$  as well. Note that in this case,  $Y(\varphi) = \lambda$ , since  $\inf_x \varphi(x)$  has no free variables (there are no  $j_1, \dots, j_n < j$ ). Let  $S = \{\nu < \lambda : \inf_x \varphi^{\mathcal{N}}(x) = 0\}$ . Since  $\mathcal{M} \equiv \mathcal{N}$ , we have  $S = \lambda$ . Now for  $k \geq 1$ ,  $S \subseteq Z(\psi_k)$ . So for  $k \geq 1$ ,  $Z(\psi_k) = \lambda$ , and thus  $Z(\psi_k) \in D'_0$ . Now if  $j > 0$ , we can assume  $n \geq 1$ , and the proof goes through as before.

Thus, by the previous lemma, there exist  $F_i \subseteq F'_j$ ,  $D_i \supseteq D'_j$  such that  $|F'_j \setminus F_j| \leq \lambda + |i|$  and  $(F_i, \emptyset, D_i)$  is  $(\lambda + |i|)$ -consistent, and  $b_j : \lambda \rightarrow N$  such that

$$\{\nu < \lambda : \psi_k^{\mathcal{N}}(b_j(\nu), b_{j_1}(\nu), \dots, b_{j_n}(\nu)) = 0\} \in D_i$$

for all  $k \geq 1$ . This shows that (1)-(5) hold when  $i$  is a successor ordinal, and so this completes the proof.  $\square$

Lastly, we give another characterization of axiomatizability for uniformly equicontinuous classes of  $L$ -structures.

**Theorem 37.** *Let  $\mathcal{C}$  be a uniformly equicontinuous class of  $L$ -structures. Then the following are equivalent:*

- (1)  $\mathcal{C}$  is axiomatizable in  $L$
- (2)  $\mathcal{C}$  is closed under isomorphisms and ultraproducts, and its complement  $\{\mathcal{M} : \mathcal{M} \text{ is an } L\text{-structure not in } \mathcal{C}\}$  is closed under ultrapowers.

The proof of this theorem is exactly the same as the proof of Theorem 5.14 in [2].

Thus, we have given two characterizations of axiomatizability for uniformly equicontinuous classes of multimetric structures.

## 12. FUTURE RESEARCH

It remains to be seen whether the class of probability spaces with two measures, each of which is absolutely continuous with respect to the other, is axiomatizable when these spaces are considered as multimetric structures. Also, the fact that the moduli of uniform continuity in Theorem 16 depend on a particular structure  $\mathcal{M}$  may have undesirable consequences for important theorems in continuous model theory other than the ones presented here. While there may be ways to avoid these consequences (for example, we may require a signature  $L$  to provide modified moduli of uniform continuity for the metric symbols of  $L$ ), we have not explored these methods yet. Lastly, we have not investigated whether the theorems proved above hold when structures are allowed to have infinitely many metrics.

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